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TECHNICAL NOTE No. 2

GENERAL INSTABILITY OF STIFFENED CIRCULAR  
CONICAL SHELLS UNDER HYDROSTATIC PRESSURE

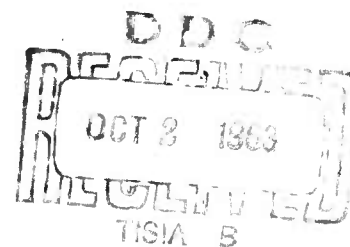
by

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Techion - Israel Institute of Technology  
Department of Aeronautical Engineering  
Haifa, Israel

TAE REPORT No. 28

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## S U M M A R Y

Donnell type equilibrium and stability equations are derived for stiffened thin conical shells. The stiffeners are considered closely spaced and are therefore assumed to be "distributed" over the whole surface of the shell. In the proposed theory the stiffeners and their spacing may vary in any prescribed manner, but here only equally spaced stiffeners are dealt with. The force - and moment - strain relations of the combined stiffener-sheet cross-section are determined by the assumption of identical normal strains at the contact surface of stiffener and sheet.

The stability equations are solved for general instability under hydrostatic pressure by the method of virtual displacements. The solution used earlier for unstiffened conical shells, which satisfies some of the boundary conditions of simple supports only approximately, is again applied here. The effect of this incomplete compliance with boundary conditions is shown to be negligible by consideration of "boundary work". The solution proposed for stiffened conical shells involves the concepts of "correcting coefficients" and minimization of corresponding "error loads".

Typical examples are analysed and the effect of eccentricity of stiffeners is investigated. Simplified approximate formulae for the critical pressure of frame-stiffened conical shells are also proposed.

## TABLE OF CONTENTS

	<i>P a g e</i>
S U M M A R Y	I
S Y M B O L S	III
1. INTRODUCTION	1
2. PREBUCKLING EQUILIBRIUM	3
3. EQUILIBRIUM DURING BUCKLING	6
4. FORCES AND MOMENTS IN STIFFENED CONICAL SHELLS	9
5. INTERNAL FORCES AND MOMENTS AS FUNCTIONS OF DISPLACEMENTS	12
6. TRUNCATED CONICAL SHELL UNDER HYDROSTATIC PRESSURE	13
7. DISPLACEMENTS	14
8. EVALUATION OF $\beta$ AND $\gamma$ BY COMPLIANCE WITH BOUNDARY CONDITIONS	15
9. MODIFICATION OF THE FIRST TWO STABILITY EQUATIONS	18
10. SOLUTION OF THE FIRST TWO "CORRECTED" STABILITY EQUATIONS	21
11. CORRECTED COEFFICIENTS	27
12. BOUNDARY CONDITIONS	33
13. COMPLIANCE WITH BOUNDARY CONDITIONS OF ZERO CIRCUMFERENTIAL DISPLACEMENT	38
14. SOLUTION	42
15. APPROXIMATE SOLUTION BY NEGLECT OF ECCENTRICITY OF THE STIFFENERS	48
16. EFFECTIVE LENGTH OF SHEET	52
17. APPROXIMATE FORMULAE FOR RING-STIFFENED SHELLS	53
18. NUMERICAL RESULTS AND DISCUSSION	57
R E F E R E N C E S	60
A C K N O W L E D G E M E N T	61

### III

#### S Y M B O L S

$a$	= distance of the top of a truncated cone from the vertex, along a generator (see Fig. 1).
$a_0$	= distance between the rings (see Fig. 1).
$a_{2n}, b_{2n}, a_{3n}, b_{3n}$	= defined by Eqs. (97).
$A_{in}, A_{rn}, B_{in}, B_{rn}$	= real displacement coefficients defined by Eq. (67).
$A_{cn}, A_{dn}$	= expressions defined by Eqs. (96).
$A_1, A_2$	= cross-sectional area of stringer or frame (ring stiffener), respectively.
$A_n, B_n$	= complex displacement coefficients defined by Eqs. (46).
$b_0$	= distance between the stringers at $x = 1$ (see Fig. 1).
$c_0$	= $\pi \rho_{av} / \ell$ , see Eq. (161).
$C_n$	= real displacement coefficient defined by Eqs. (46).
$d_{01} \dots d_{11} \dots d_{21} \dots$	= defined by Eqs. (76) to (78).
$D$	= $Eh^3 / 12 (1 - \nu^2)$
$e_1$	= distance of the centroid of the stringer cross section from the shell middle surface (see Fig. 1).
$e_2$	= distance of the centroid of the ring cross section from the shell middle surface (see Fig. 1).

# IV

$E, E'_1, E'_2$	= moduli of elasticity of sheet or stiffeners, respectively.
$E_1, E_2$	= effective moduli of elasticity of stiffeners (see Section 16).
$F_1^A(n) \dots R(n, m)$	= expressions defined by Eqs. (128) to (142).
$g_{in}, g_{jn}$	= defined by Eq. (115).
$g(\psi)$	= $p_{cr}/\bar{p}_{cr}$ , defined by Eq. (159).
$G_1, G_2$	= shear moduli of the stiffeners.
$h$	= thickness of shell.
$I_{01}, I_{02}$	= moment of inertia of stringer or frame cross-section respectively, about the line of reference (the middle line of the sheet).
$I_{11}, I_{22}$	= moment of inertia of stringer or frame cross-section, respectively, about their centroidal axis.
$I_{t1}, I_{t2}$	= torsion constants of stiffener cross-section.
$I_1^K(n,m), I_2^K(n,m), I_3^K(n,m)$	= expressions defined by Eqs. (98).
$k_1$ to $k_6, k_{56}$	= correcting coefficients defined by Eqs. (87) to (92).
$k_u, k_o, k_m$	= spring constants defined by Eqs. (106).
$\ell$	= $a(x_2 - 1)$ , see Eqs. (161).
$n, m$	= integers.
$M_{x0} \dots N_{x0} \dots$	= moments and forces prior to buckling.
$M_x \dots N_x \dots$	= additional moments and forces caused by buckling.



$M_x', \dots, N_x'$	= total moments and forces during buckling.
$\bar{M}_{x0}, \dots, \bar{N}_{x0}, \dots$	= moments and forces acting at the boundaries prior to buckling.
$N$	= number of displacement terms.
$M_0(n), \dots, N_0(n), \dots$	= expressions defined by Eqs. (69).
$p, p_{cr}, p_0$	= hydrostatic pressure, critical pressure of stiffened conical shell and critical pressure of unstiffened conical shell, respectively.
$\bar{p}_{cr}$	= critical pressure of equivalent cylindrical shell.
$q_x, q_\phi, q_z$	= external loads.
$R_1, R_2$	= radii of small or large end of truncated cone, respectively.
$s$	= complex number, $s = \gamma + i n \beta$
$S$	= $t_0^2 + c_0^2$ , see Eq. (163).
$t$	= number of circumferential waves.
$t_0$	= $t/\cos \alpha$ , see Eq. (161)
$T(n, m)$	= expression defined by Eqs. (117) and (126).
$u^*$	= displacement along a generator
$u$	= non-dimensional displacement along a generator = $u^*/a$
$U_0$	= total potential energy prior to buckling.
$U$	= additional potential energy caused by buckling.

# VI

$v^*$	= circumferential displacement
$v$	= non-dimensional circumferential displacement = $v^*/a$
$w^*$	= radial displacement.
$w$	= non-dimensional radial displacement = $w^*/a$
$x^*$	= axial coordinate along a generator.
$x$	= non-dimensional axial coordinate = $x^*/a$
$x_2$	= ratio of the distance of the bottom of a truncated cone from the vertex, to that of the top.
$z^*$	= radial coordinate.
$z_1 z_2$	= distance of the centroid of the stringer-shell, or ring-shell combination from the middle surface (see Fig. 1).
$\alpha$	= cone angle
$\beta$	= $\pi / \ln_e x_2$ , see Eq. (50).
$\gamma$	= defined by Eq. (56), = $(1/2) [1 - \nu / (1 + \eta_{01})]$
$\epsilon_x, \epsilon_\phi, \gamma_{x\phi}$	= middle surface strains.
$\zeta_1$	= $E_1 A_1 e_1 a / b_0 D$
$\zeta_2$	= $E_2 A_2 e_2 a / a_0 D$
$\eta_{01}$	= $E_1 I_{01} / b_0 D$

# VII

$\eta_{02}$	$= E_2 I_{02}/a_0 D$
$\eta_{t1}$	$= G_1 I_{t1}/b_0 D$
$\eta_{t2}$	$= G_2 I_{t2}/a_0 D$
$\eta_1$	$= 12(1-\nu^2)(E_1/E)\{I_{11}/b_0 h^3 + (A_1/b_0 h)[(e_1 - \bar{z}_1)/h]^2\} + 12(z_1/h)^2 \bar{x}$ , Eq.(155).
$\eta_2$	$= 12(1-\nu^2)(E_2/E)\{I_{22}/a_0 h^3 + (A_2/a_0 h)[(e_2 - \bar{z}_2)/h]^2\} + 12(\bar{z}_2/h)^2$ , Eq.(147).
$\kappa_x^*, \kappa_\phi^*, \kappa_{x\phi}^*$	$=$ changes of curvature and twist of the middle surface.
$\kappa_x, \kappa_\phi, \kappa_{x\phi}$	$=$ non-dimensional changes of curvature and twist of the middle surface $\kappa_x = a \kappa_x^*, \kappa_\phi = a \kappa_\phi^*, \kappa_{x\phi} = a \kappa_{x\phi}^*$
$\lambda_p$	$= p a^3 \tan \alpha/D$
$\mu_1$	$= (1-\nu^2) E_1 A_1/E b_0 h$
$\mu_2$	$= (1-\nu^2) E_2 A_2/E a_0 h$
$\mu_{11}, \chi_{12}, \chi_{23}$ to $\chi_{26}$	$=$ defined by Eqs. (64).
$\nu$	$=$ Poisson's ratio.
$\phi$	$=$ circumferential coordinate.
$\chi_1$	$= (1-\nu^2) E_1 A_1 e_1/E b_0 h a$
$\chi_2$	$= (1-\nu^2) E_2 A_2 e_2/E a_0 h a$

Subscripts following a comma indicate differentiation.

## I. I N T R O D U C T I O N

In order to increase the resistance of shells to buckling, they are strengthened by stiffeners. In this manner, the critical load can be increased several times by only little addition of material. It is assumed that the buckling is of the general instability type, that is, the shell and its stiffeners buckle together.

The stiffeners are considered closely spaced, and are therefore assumed to be "distributed" over the whole surface of the shell. In the proposed theory, the stiffeners need not be equal and equally spaced, but may change in any prescribed manner. The present report, however, deals with conical shells stiffened by equal and equally spaced frames (rings) and stringers. This is the usual way of stiffening, though not necessarily the optimal one. There may be some other law of stiffener distribution which would yield maximum stiffening for a given addition of material. This optimization problem is not considered here, but could be solved by the proposed method of solution.

The relations between the strains and the internal forces and moments of the combined stiffener-sheet section are found by the assumption that the normal strains, in the stiffener and in the sheet, are equal at their point of contact. Thus, the eccentricity of the stiffeners relative to the sheet is taken into account. The analysis permits the sheet and the stiffeners to be of different materials.

The middle surface of the shell (without stiffeners) is taken as the surface of reference. The stress-strain relations in this surface are assumed as in an unstiffened shell.

It is assumed that for general instability a stiffened conical shell buckles in a mode similar to that of an unstiffened conical shell. Hence, the displacements which were used for unstiffened conical shells (Ref. 1) could be applied to the problem of buckling of a stiffened conical shell. These displacements satisfy some of the assumed boundary conditions of simple supports only approximately, and imply fictitious elastic restraints. The effect of these restraints was, however, shown to be negligible (Ref. 2). Here, another method is proposed to estimate the effect of the partial fulfillment only of these boundary conditions upon the value of the critical load. The method is based on consideration of the "boundary-work" – the work done by the internal forces and moments at the boundaries.

The method of virtual displacements, used for solution of the present problem, stems from the principle of virtual work. It only requires that the displacements fulfil geometrical boundary conditions. The displacements used here fulfil rigorously the boundary conditions of zero radial displacement, while fulfilling only approximately the boundary condition of zero displacement in the circumferential direction. A method of satisfying this boundary condition in general, although not for every term of the displacement series, is then proposed.

For unstiffened conical shells the displacements solve the first two stability equations exactly. In the case of stiffened conical shells, however, these equations are not solved exactly by them. This occurs on account of the additional terms introduced by the frames and stringers. But the same displacements may be used in the following manner. In the first two stability equations, the terms which do not lend themselves to solution, are replaced by terms which do. The latter are multiplied by coefficients called "correcting coefficients". Thus, "corrected" stability equations, which can be solved exactly by the displacements, are obtained. The original terms, removed from the first stability equations, and the "correcting" terms, which replace them there, with opposite sign, are added. These sums are called "error-loads". The "correcting coefficients" are calculated by equating the virtual work done by the "error-load" to zero.

In the analysis the "effective sheet length" is considered as a reduction in the moduli of elasticity of the stiffeners.

A simpler approximate method of calculation is obtained by neglecting the eccentricity of the stiffeners. Then, "correcting coefficients" are not-needed, and the calculations become easier.

A simple approximate formula for calculation of the critical pressure of a frame-stiffened conical shell, by consideration of an equivalent cylindrical shell, is proposed. This formula is based on a similar one for unstiffened conical shells.

Some typical cases are calculated by the above methods. It is shown that the effect of the "boundary work" upon the value of the critical load is small. Frames (rings) increase the resistance of the shell against buckling, under hydrostatic pressure, considerably. It is shown that the placing of the frames is of importance. Frames on the inside of the shell yield higher general instability loads than frames on the outside. Stringers are much less effective in stiffening of shells under hydrostatic pressure,

and the effect of their eccentricity is opposite; outside stringers yield higher critical loads than inside stringers.

## 2. PREBUCKLING EQUILIBRIUM

The strain-displacement relations at the middle surface of a deformed conical shell used in the derivation are those given by Love (Ref. 3).

$$\begin{aligned}\epsilon_x &= u_{,x} \\ \epsilon_\phi &= v_{,\phi}/x \sin \alpha + u/x - w \cot \alpha/x \\ \gamma_{x\phi} &= v_{,x} - v/x + u_{,\phi}/x \sin \alpha\end{aligned}\tag{1}$$

The curvatures are defined as

$$\begin{aligned}\kappa_x &= w_{,xx} \\ \kappa_\phi &= w_{,x}/x + w_{,\phi\phi}/x^2 \sin^2 \alpha \\ \kappa_{x\phi} &= w_{,x\phi}/x \sin \alpha - w_{,\phi}/x^2 \sin \alpha\end{aligned}\tag{2}$$

These are the curvature displacement relations obtained by Seide (Ref. 4) by omitting the terms involving the circumferential displacement  $v$  from Love's definition, on account of their negligible effect in cylindrical shells and vanishing in the case of circular plates.

The analysis is written in non-dimensional form, and the non-dimensional distances, displacements and curvatures are defined by the equations:

$$\begin{aligned}x &= x^*/a & \kappa_x &= a \kappa_x^* \\ z &= z^*/a & \kappa_\phi &= a \kappa_\phi^* \\ u &= u^*/a & \kappa_{x\phi} &= a \kappa_{x\phi}^* \\ v &= v^*/a \\ w &= w^*/a\end{aligned}\tag{3}$$

Virtual displacements are applied to a segment of the shell, which is in equilibrium. Hence, by the principle of virtual work (the virtual work done by the stresses must be equal to the virtual work done by the external forces and moments),

$$\begin{aligned}
 0 = \delta U_0 = & \int_{\phi_1}^{\phi_2} \int_{x_1}^{x_2} [N_{x0} \delta \epsilon_x + N_{\phi 0} \delta \epsilon_\phi + N_{x\phi 0} \delta \gamma_{x\phi} - M_{x0} \delta (\kappa_x/a) - M_{\phi 0} \delta (\kappa_\phi/a) \\
 & + M_{x\phi 0} \delta (\kappa_{x\phi}/a) - M_{\phi x 0} \delta (\kappa_{x\phi}/a) - (q_x a \delta u \\
 & + q_\phi a \delta v + q_x a \delta w)] a^2 x \sin \alpha \, dx \, d\phi \\
 & - \int_{\phi_1}^{\phi_2} \{ x a [\bar{N}_{x0} \delta (a u) + \bar{N}_{x\phi 0} \delta (a v) - \bar{M}_{x0} \delta (w_{,x}) + \bar{M}_{x\phi 0} \delta (w_{,\phi})/x \sin \alpha \\
 & + \bar{Q}_{x0} \delta (a w)] \}_{x=x_1}^{x=x_2} \sin \alpha \, d\phi \\
 & - \int_{x_1}^{x_2} [\bar{N}_{\phi 0} \delta (a v) + \bar{N}_{\phi x 0} \delta (a u) - \bar{M}_{\phi 0} \delta (w_{,\phi})/x \sin \alpha - \bar{M}_{\phi x 0} \delta (w_{,x}) \\
 & + \bar{Q}_{\phi 0} \delta (a w)]_{\phi=\phi_1}^{\phi=\phi_2} a \, dx
 \end{aligned} \tag{4}$$

where the index zero indicates the state of the shell before buckling, and the barred quantities are the external forces and moments acting on the boundaries.

It is assumed that  $N_{x\phi}$  is carried only by the sheet (i.e. the stiffeners do not transmit shear). Hence, it is assumed as in Ref. 4 that

$$N_{x\phi} = N_{\phi x}$$

The work done by the internal forces  $Q_x$  and  $Q_y$  is neglected in the expression of the virtual work, as the theory developed is a Donnell type theory.

Substitution of Eqs. (1) and (2) into Eq. (4), and integration by parts yields;

$$0 = \delta U_0 = - \int_{\phi_1}^{\phi_2} \int_{x_1}^{x_2} \{ [(x N_{x0})_{,x}/a x - N_{\phi 0}/a x + N_{x\phi 0,\phi}/a x \sin \alpha + q_x] \delta u$$

$$\begin{aligned}
& + [N_{\phi_0, \phi} / a x \sin \alpha + (x^2 N_{x\phi_0})_{,x} / a x^2 + q_\phi] \delta v \\
& + [(x M_{x_0})_{,xx} / a^2 x - M_{\phi_0, x} / a^2 x + M_{\phi_0, \phi\phi} / a^2 x^2 \sin^2 \alpha \\
& - (x M_{x\phi_0})_{,x\phi} / a^2 x^2 \sin \alpha + (x M_{\phi x_0})_{,x\phi} / a^2 x^2 \sin \alpha \\
& + N_{\phi_0} \cot \alpha / a x + q_x] \delta w \} a^3 x \sin \alpha \, dx d\phi \\
& + \int_{\phi_1}^{\phi_2} \{ a x [(N_{x_0} - \bar{N}_{x_0}) a \delta u + (N_{x\phi_0} - \bar{N}_{x\phi_0}) a \delta v - (M_{x_0} - \bar{M}_{x_0}) \delta(w_{,x})] \\
& + [(x M_{x_0})_{,x} - M_{\phi_0} - M_{x\phi_0, \phi} / \sin \alpha + M_{\phi x_0, \phi} / \sin \alpha - a x \bar{Q}_{x_0} \\
& + \bar{M}_{x\phi_0, \phi} / \sin \alpha] a \delta w \} \Big|_{x=x_1}^{x=x_2} \sin \alpha \, d\phi \\
& + \int_{x_1}^{x_2} \{ (N_{\phi_0} - \bar{N}_{\phi_0}) a \delta v + (N_{\phi x_0} - \bar{N}_{\phi x_0}) a \delta u - (M_{\phi_0} - \bar{M}_{\phi_0}) \delta(w_{, \phi}) / x \sin \alpha \\
& + [M_{\phi_0, \phi} / x a \sin \alpha - (x M_{x\phi_0})_{,x} / a x + (x M_{\phi x_0})_{,x} / a x - \bar{Q}_{\phi_0} \\
& - \bar{M}_{\phi x_0, x} / a] a \delta w \} \Big|_{\phi=\phi_1}^{\phi=\phi_2} a \, dx \\
& + (M_{x\phi_0} - \bar{M}_{x\phi_0}) a \delta w \Big|_{x=x_1}^{x=x_2} \Big|_{\phi=\phi_1}^{\phi=\phi_2} - (M_{\phi x_0} - \bar{M}_{\phi x_0}) a \delta w \Big|_{x=x_1}^{x=x_2} \Big|_{\phi=\phi_1}^{\phi=\phi_2}
\end{aligned} \tag{6}$$

Eq. (6) yields the following equilibrium equations and boundary conditions:

In the shell

$$(x N_{x_0})_{,x} / a x - N_{\phi_0} / a x + N_{x\phi_0, \phi} / a x \sin \alpha + q_x = 0 \tag{7}$$

$$N_{\phi_0, \phi} / a x \sin \alpha + (x^2 N_{x\phi_0})_{,x} / a x^2 + q_\phi = 0 \tag{8}$$

$$\begin{aligned}
& (x M_{x_0})_{,xx} / a^2 x - M_{\phi_0, x} / a^2 x + M_{\phi_0, \phi\phi} / a^2 x^2 \sin^2 \alpha \\
& - (x M_{x\phi_0})_{,x\phi} / a^2 x^2 \sin \alpha + (x M_{\phi x_0})_{,x\phi} / a^2 x^2 \sin \alpha + N_{\phi_0} \cot \alpha / a x + q_x = 0
\end{aligned} \tag{9}$$

Along the circles  $x = x_1$  and  $x = x_2$



$$N_{x0} = \bar{N}_{x0} \quad N_{x\phi 0} = \bar{N}_{x\phi 0} \quad M_{x0} = \bar{M}_{x0} \quad (10)$$

$$(x M_{x0})_{,x} - M_{\phi 0} - M_{x\phi 0,\phi} / \sin \alpha + M_{\phi x 0,\phi} / \sin \alpha = a x \bar{Q}_{x0} - \bar{M}_{x\phi 0,\phi} / \sin \alpha \quad (11)$$

Along the generators  $\phi = \phi_1$  and  $\phi = \phi_2$

$$N_{\phi 0} = \bar{N}_{\phi 0} \quad N_{\phi x 0} = \bar{N}_{\phi x 0} \quad M_{\phi 0} = \bar{M}_{\phi 0} \quad (12)$$

$$M_{\phi 0,\phi} / a \sin \alpha - (x M_{x\phi 0})_{,x} / a x + (x M_{\phi x 0})_{,x} / a x = \bar{Q}_{\phi 0} + \bar{M}_{\phi x 0,x} / a \quad (13)$$

and at the corners of the segment

$$M_{x\phi 0} = \bar{M}_{x\phi 0} \quad M_{\phi x 0} = \bar{M}_{\phi x 0} \quad (14)$$

The geometrical boundary conditions are not discussed here, as it has been assumed in the derivation that the forces acting on the boundaries obey any given geometrical boundary conditions.

### 3. EQUILIBRIUM DURING BUCKLING

The equilibrium at buckling is obtained by consideration of the additional virtual work during buckling. The displacements are now the additional displacements caused by buckling, and the prebuckling displacements are assumed to be small so that also the additional displacements can be related to the undeformed geometry of the shell.

The stretching of the middle surface introduces the following nonlinear terms, which have to be added to the strains of Eqs. (1),

$$\begin{aligned} \epsilon'_x &= (w_{,x})^2 / 2 \\ \epsilon'_\phi &= (w_{,\phi})^2 / 2 x^2 \sin^2 \alpha \\ \gamma'_{x\phi} &= w_{,x} w_{,\phi} / x \sin \alpha \end{aligned} \quad (15)$$

The connection between forces and moments prior and during buckling is

$$\begin{aligned} N' &= N_0 + N \\ M' &= M_0 + M \end{aligned} \quad (16)$$

Where  $N'$  and  $M'$  are the total forces and moments during buckling,  $N_0$  and  $M_0$  are those prior to buckling, and  $N$  and  $M$  are the additional forces and moments caused by buckling.

Now, if to the shell is given virtual displacements, the virtual work  $\delta U$  done during buckling must also vanish, since the shell is in a state of equilibrium. The internal forces and moments  $N_0$  and  $M_0$  in Eqs. (4) must be replaced by  $N'$  and  $M'$  from Eqs. (16), and the nonlinear virtual work done by the membrane forces prior to buckling  $N_{x0}$ ,  $N_{\phi 0}$  and  $N_{x\phi 0}$ ,

$$\Delta \delta U = \int_{\phi_1}^{\phi_2} \int_{x_1}^{x_2} (N_{x0} \delta \epsilon'_x + N_{\phi 0} \delta \epsilon'_\phi + N_{x\phi 0} \delta \gamma'_{x\phi}) x a^2 \sin \alpha dx d\phi \quad (17)$$

must be added.

After Eqs. (7) to (14) are also taken into account, the expression for the virtual work during buckling becomes:

$$\begin{aligned} 0 = \delta U = & - \int_{\phi_1}^{\phi_2} \int_{x_1}^{x_2} \{ [(xN_x)_{,x}]/ax - N_\phi/ax + N_{x\phi,\phi}/ax \sin \alpha \} \delta u + [N_{\phi,\phi}/ax \sin \alpha, \\ & + (x^2 N_{x\phi})_{,x}/ax^2 \} \delta v + [(xM_x)_{,xx}/a^2 x - M_{\phi,x}/a^2 x \\ & + M_{\phi,\phi\phi}/a^2 x^2 \sin^2 \alpha - (xM_{x\phi})_{,x\phi}/a^2 x^2 \sin \alpha + (xM_{\phi x})_{,x\phi}/a^2 x^2 \sin \alpha \\ & + N_\phi \cot \alpha/ax + (xN_{x0} w_{,x})_{,x}/ax + (N_{\phi 0} w_{,\phi})_{,\phi}/ax^2 \sin^2 \alpha \\ & + (N_{x\phi 0} w_{,x})_{,\phi}/ax \sin \alpha + (N_{x\phi 0} w_{,\phi})_{,x}/ax \sin \alpha \} \delta w \} a^3 \sin \alpha dx d\phi \\ & + \int_{\phi_1}^{\phi_2} \{ ax [N_x a \delta u + N_{x\phi} a \delta v - M_x \delta (w_{,x})] + [(xM_x)_{,x} - M_\phi - M_{x\phi,\phi}/\sin \alpha \\ & + M_{\phi x,\phi}/\sin \alpha + N_{x0} ax w_{,x} + N_{x\phi 0} a w_{,\phi}/\sin \alpha] a \delta w \} \Big|_{x=x_1}^{x=x_2} \sin \alpha d\phi \end{aligned} \quad (18)$$

$$\begin{aligned}
& + \int_{x_1}^{x_2} \{ N_\phi a \delta v + N_{\phi x} a \delta u - M_\phi \delta(w, \phi) / x \sin \alpha + [M_{\phi, \phi} / a x \sin \alpha \\
& - (x M_{x\phi})_{,x} / a x + (x M_{\phi x})_{,x} / a x + N_{\phi 0} w, \phi / x \sin \alpha + N_{x\phi 0} w, x] a \delta w \} \Big|_{\phi=\phi_1}^{\phi=\phi_2} a dx \\
& + M_{x\phi} a \delta w \Big|_{x=x_1}^{x=x_2} \Big|_{\phi=\phi_1}^{\phi=\phi_2} - M_{\phi x} a \delta w \Big|_{x=x_1}^{x=x_2} \Big|_{\phi=\phi_1}^{\phi=\phi_2}
\end{aligned}$$

Hence, the following stability equations and boundary conditions are obtained:

$$(x N_x)_{,x} / a x - N_\phi / a x + N_{x\phi, \phi} / a x \sin \alpha = 0 \quad (19)$$

$$N_{\phi, \phi} / a x \sin \alpha + (x^2 N_{x\phi})_{, \phi} / a x^2 = 0 \quad (20)$$

$$\begin{aligned}
& (x M_x)_{,xx} / a^2 x - M_{\phi, x} / a^2 x + M_{\phi, \phi \phi} / a^2 x^2 \sin^2 \alpha - (x M_{x\phi})_{,x\phi} / a^2 x^2 \sin \alpha \\
& + (x M_{\phi x})_{,x\phi} / a^2 x^2 \sin \alpha + N_\phi \cot \alpha / a x + (x N_{x0} w, x)_{,x} / a x \\
& + (N_{\phi 0} w, \phi)_{, \phi} / a x^2 \sin^2 \alpha + (N_{x\phi 0} w, x)_{, \phi} / a x \sin \alpha + (N_{x\phi 0} w, \phi)_{, x} / a x \sin \alpha = 0
\end{aligned} \quad (21)$$

Along  $x = x_1$  and  $x = x_2$

$$N_x = 0 \quad \text{or} \quad u = 0 \quad (22)$$

$$N_{x\phi} = 0 \quad \text{or} \quad v = 0 \quad (23)$$

$$(x M_x)_{,x} - M_\phi - M_{x\phi, \phi} / \sin \alpha + M_{\phi x, \phi} / \sin \alpha + N_{x0} x a w, x + a N_{x\phi 0} w, \phi / \sin \alpha = 0 \quad (24)$$

$$\text{or} \quad w = 0$$

$$M_x = 0 \quad \text{or} \quad w, x = 0 \quad (25)$$

Along  $\phi = \phi_1$  and  $\phi = \phi_2$

$$N_\phi = 0 \quad \text{or} \quad v = 0 \quad (26)$$

$$N_{\phi x} = 0 \quad \text{or} \quad u = 0 \quad (27)$$

$$M_{\phi, \phi} / ax \sin \alpha - (x M_{x\phi})_{,x} / ax + (x M_{\phi x})_{,x} / ax + N_{\phi 0} w_{,\phi} / x \sin \alpha + N_{x\phi 0} w_{,x} = 0$$

$$\text{or} \quad w = 0 \quad (28)$$

$$M_{\phi} = 0 \quad \text{or} \quad w_{,\phi} = 0 \quad (29)$$

and at the corners of the segment

$$M_{x\phi} = 0$$

$$\text{or} \quad w = 0 \quad (30)$$

$$M_{\phi x} = 0$$

#### 4. FORCES AND MOMENTS IN STIFFENED CONICAL SHELLS

In the cross-section of the shell the strain is assumed to vary linearly as in Ref. 5,

$$\epsilon_x(z^*) = \epsilon_x - z^* \kappa_x / a$$

$$\epsilon_{\phi}(z^*) = \epsilon_{\phi} - z^* \kappa_{\phi} / a \quad (31)$$

$$\gamma_{x\phi}(z^*) = \gamma_{x\phi} - 2z^* \kappa_{x\phi} / a$$

where  $z^*$  is the physical coordinate.

The stress-strain relations are:

$$\sigma_x(z) = [E / (1 - \nu^2)] [\epsilon_x(z) + \nu \epsilon_{\phi}(z)]$$

$$\sigma_{\phi}(z) = [E / (1 - \nu^2)] [\epsilon_{\phi}(z) + \nu \epsilon_x(z)] \quad (32)$$

$$\tau_{x\phi}(z) = [E / 2(1 + \nu)] \gamma_{x\phi}(z)$$

if the stresses perpendicular to the surface of the shell are neglected.

For the stiffeners the following assumptions are made:

1. The stiffeners are „distributed” over the whole surface of the shell.
2. The normal strains  $\epsilon_x(z)$  and  $\epsilon_\phi(z)$  vary linearly also in the stiffener according to equations (31). The normal strains in the stiffener and in the sheet are equal at their point of contact.
3. The stiffeners do not transmit shear. The shear membrane force  $N_{x\phi}$  is carried entirely by the sheet.
4. The torsional rigidity of the stiffener cross-section is added to that of the sheet. (The actual increase in torsional rigidity is larger than that assumed.)

Substitution of Eqs. (31) into (32), and taking into account the above assumption, yields the following expressions for the normal stresses:

in the sheet

$$\begin{aligned}\sigma_x(z) &= [E/(1 - \nu^2)] [\epsilon_x - z^* \kappa_x / a + \nu (\epsilon_\phi - z^* \kappa_\phi / a)] \\ \sigma_\phi(z) &= [E/(1 - \nu^2)] [\epsilon_\phi - z^* \kappa_\phi / a + \nu (\epsilon_x - z^* \kappa_x / a)]\end{aligned}\quad (33)$$

and in the stiffeners

$$\begin{aligned}\sigma_x(z) &= E_1 \epsilon_x(z) = E_1 (\epsilon_x - z^* \kappa_x / a) \\ \sigma_\phi(z) &= E_2 \epsilon_\phi(z) = E_2 (\epsilon_\phi - z^* \kappa_\phi / a)\end{aligned}\quad (34)$$

where  $E_1$  and  $E_2$  are the “effective” moduli of elasticity of stringers and frames, respectively, defined in Section 17.

The membrane force  $N_x$  per unit length is:

$$\begin{aligned}N_x &= \int \sigma_x dz^* = \int_{-h/2}^{+h/2} \sigma_x dz^* + (1/b_0 x) \int_{A_1} \sigma_x dA_1 \\ &= \int_{-h/2}^{+h/2} [E/(1 - \nu^2)] [(\epsilon_x - z^* \kappa_x / a) + \nu (\epsilon_\phi - z^* \kappa_\phi / a)] dz^* + (1/b_0 x) \int_{A_1} E_1 (\epsilon_x - z^* \kappa_x / a) dA_1\end{aligned}$$

$$= [Eh/(1-\nu^2)] (\epsilon_x + \nu \epsilon_\phi) + E_1 A_1 \epsilon_x / b_0 x - E_1 e_1 A_1 \kappa_x / b_0 x a \quad (35)$$

All the forces and the moments about the line of reference (the middle surface of the sheet) are obtained similarly:

$$\begin{aligned} N_x &= [Eh/(1-\nu^2)] [\epsilon_x (1+\mu_1/x) + \nu \epsilon_\phi - \chi_1 \kappa_x / x] \\ N_\phi &= [Eh/(1-\nu^2)] [\epsilon_\phi (1+\mu_2) + \nu \epsilon_x - \chi_2 \kappa_\phi] \end{aligned} \quad (36)$$

$$N_x \phi = N_{\phi x} = Eh \gamma_{x\phi} / 2(1+\nu)$$

$$\begin{aligned} M_x &= (-D/a)[(1 + \eta_{01}/x)\kappa_x + \nu \kappa_\phi - \zeta_1 \epsilon_x / x] \\ M_\phi &= (-D/a)[(1 + \eta_{02})\kappa_\phi + \nu \kappa_x - \zeta_2 \epsilon_\phi] \\ M_x \phi &= (+D/a)[(1 - \nu) + \eta_{t1}/x] \kappa_x \phi \\ M_{\phi x} &= (-D/a)[(1 - \nu) + \eta_{t2}] \kappa_x \phi \end{aligned} \quad (37)$$

Where  $\mu_1/x$  and  $\mu_2$  are the increases in effective cross-sectional area of the shell due to stringers and frames respectively, defined by

$$\begin{aligned} \mu_1 &= (1 - \nu^2) E_1 A_1 / E b_0 h \\ \mu_2 &= (1 - \nu^2) E_2 A_2 / E a_0 h \end{aligned} \quad (38)$$

$\chi_1/x$  and  $\chi_2$  are the changes in extensional stiffness caused by the eccentricities of stringers and frames respectively, defined by

$$\begin{aligned} \chi_1 &= (1 - \nu^2) E_1 A_1 e_1 / E b_0 h a \\ \chi_2 &= (1 - \nu^2) E_2 A_2 e_2 / E a_0 h a \end{aligned} \quad (39)$$

$\eta_{01}/x$ ,  $\eta_{02}$ ,  $\eta_{t1}/x$  and  $\eta_{t2}$  are the increases in bending and twisting stiffness of the shell due to stringers and frames respectively, defined by

$$\begin{aligned}\eta_{01} &= E_1 I_{01} / b_0 D \\ \eta_{02} &= E_2 I_{02} / a_0 D \\ \eta_{t1} &= G_1 I_{t1} / b_0 D \\ \eta_{t2} &= G_2 I_{t2} / a_0 D\end{aligned}\tag{40}$$

and  $\zeta_1/x$  and  $\zeta_2$  are the changes in bending stiffness caused by the eccentricities of stringers and frames respectively, defined by

$$\begin{aligned}\zeta_1 &= E_1 A_1 e_1 a / b_0 D \\ \zeta_2 &= E_2 A_2 e_2 a / a_0 D\end{aligned}\tag{41}$$

Nothing has yet been said about the manner of distribution of the stiffeners and their magnitude. The distance between the frames can be some function of  $x$ , and the magnitude of the stiffeners can be a function of  $x$  and  $\phi$ . The changes in the stiffnesses in Eqs. (36) and (37) are then functions of  $x$  and  $\phi$ . This does not cause fundamental difficulties if these functions are known from the beginning.

In the following analysis, the frames are equal and equally spaced, and the stringers are equal. The distances between the stringers are linear functions of  $x$  ( $b = b_0 x$ ), but this has already been taken into account in Eqs. (36) and (37). Hence, all the values in Eqs. (38) to (41) are constant.

## 5. INTERNAL FORCES AND MOMENTS AS FUNCTIONS OF DISPLACEMENTS

Substitution of Eqs. (1) and (2) into Eqs. (36) and (37) yields the additional internal forces and moments acting during buckling as functions of the additional displacements:

$$N_x = [Eh/(1 - \nu^2)] [(1 + \mu_1/x)u_{,x} + \nu(\nu_{,\phi}/x \sin a + u/x - w \cot a/x) - \chi_1 w_{,xx}/x]$$

$$\begin{aligned}
N_{\phi} &= [Eh/(1 - \nu^2)] [(1 + \mu_2)(v_{,\phi}/x \sin \alpha + u/x - w \cot \alpha/x) + \nu u_{,x} - \chi_2(w_{,x}/x \\
&\quad + w_{,\phi\phi}/x^2 \sin^2 \alpha)] \\
N_{x\phi} &= [Eh/2(1 + \nu)] [v_{,x} - v/x + u_{,\phi}/x \sin \alpha]
\end{aligned} \tag{42}$$

$$\begin{aligned}
M_x &= [-D/a] [(1 + \eta_{01}/x) w_{,xx} + \nu(w_{,x}/x + w_{,\phi\phi}/x^2 \sin^2 \alpha) - \zeta_1 u_{,x}/x] \\
M_{\phi} &= [-D/a] [(1 + \eta_{02})(w_{,x}/x + w_{,\phi\phi}/x^2 \sin^2 \alpha) + \nu w_{,xx} \\
&\quad - \zeta_2(v_{,\phi}/x \sin \alpha + u/x - w \cot \alpha/x)] \\
M_{x\phi} &= [+D/a] [(1 - \nu) + \eta_{t1}/x] (w_{,x\phi}/x \sin \alpha - w_{,\phi}/x^2 \sin \alpha) \\
M_{\phi x} &= [-D/a] [(1 - \nu) + \eta_{t2}] (w_{,x\phi}/x \sin \alpha - w_{,\phi}/x^2 \sin \alpha)
\end{aligned} \tag{43}$$

## 6. TRUNCATED CONICAL SHELL UNDER HYDROSTATIC PRESSURE.

A "simply supported" circular truncated conical shell is considered, closed at the ends by bulkheads which offer no restraint against displacement or rotation of the generators of the shell while being rigid perpendicular to them.

The load is uniform hydrostatic pressure acting on the shell and the bulkheads. It is assumed that the stress state prior to buckling is represented satisfactorily by the membrane stresses

$$\begin{aligned}
N_{x0} &= -(p/2)ax \tan \alpha \\
N_{\phi 0} &= -pax \tan \alpha \\
N_{x\phi 0} &= 0
\end{aligned} \tag{44}$$



In the complete truncated conical shell the boundary conditions along the lines  $\phi_1 = 0$  and  $\phi_2 = 2\pi$  (which are the same line) are satisfied automatically, and the shell has no corners. For the case of hydrostatic pressure, Eq. (18) becomes then

$$\begin{aligned} \delta U = & - \int_0^{2\pi} \int_1^{x_2} \{ [(xN_x)_{,x}/ax - N_{\phi}/ax + N_{x\phi,x}/ax \sin \alpha] \delta u + [N_{\phi,\phi}/ax \sin \alpha \\ & + (x^2 N_{x\phi})_{,x}/ax^2] \delta v + [(xM_x)_{,xx}/a^2 x - M_{\phi,x}/a^2 x + M_{\phi,\phi\phi}/ax^2 \sin^2 \alpha \\ & - (xM_{x\phi})_{,x\phi}/a^2 x^2 \sin \alpha + (xM_{\phi x})_{,x\phi}/a^2 x^2 \sin \alpha + N_{\phi} \cot \alpha / ax \\ & - p \tan \alpha (xw_{,xx}/2 + w_{,x} + w_{,\phi\phi}/x \sin^2 \alpha) \} \delta w \} a^3 x \sin \alpha dx d\phi \\ & + \int_0^{2\pi} \{ ax [N_x a \delta u + N_{x\phi} a \delta v - M_x \delta (w_{,x})] + [(xM_x)_{,x} - M_{\phi} - M_{x\phi,\phi}/\sin \alpha \\ & + M_{\phi x,\phi}/\sin \alpha - (p/2) a^2 x^2 \tan \alpha w_{,x}] a \delta w \} \Big|_{x=1}^{x=x_2} \sin \alpha d\phi = 0 \end{aligned} \quad (45)$$

The critical pressure is obtained from the above condition that the virtual work must vanish. After substitution of Eqs. (42) and (43), Eq. (45) becomes a function of the displacements only. Thus, the problem reduces to finding displacements  $u$ ,  $v$ , and  $w$  which satisfy Eq. (45).

## 7. D I S P L A C E M E N T S.

The admissible displacements used in the solution of Eq. (45) are assumed as in Refs. 1, 6 and 7 in the form

$$\begin{aligned} u &= \Re \sum_{n=1}^N A_n x^n \sin t\phi \\ v &= \Re \sum_{n=1}^N B_n x^n \cos t\phi \end{aligned}$$

$$w = \Im \sum_{n=1}^N C_n x^s \sin t \phi \quad (46)$$

Where  $\Im$  indicates the imaginary part of the expressions,  $A_n$  and  $B_n$  are complex coefficients and  $C_n$  and  $t$  are real. ( $t$  is the number of circumferential waves of the buckling deformation),  $s$  is the complex number

$$s = \gamma + i\beta \quad (47)$$

$\gamma$  and  $\beta$  are real, and are calculated from the boundary conditions,  $n$  is a real integer, and  $i = \sqrt{-1}$ .

The displacements of Eqs. (46) solve the first two stability equations of an unstiffened conical shell exactly, and the third stability equation is solved in Ref. 1 by the Galerkin method.

In the case of a stiffened conical shell these displacements do not solve the first two stability equations exactly. Nevertheless, since it is assumed that the form of buckling in general instability of a stiffened conical shell does not differ much from that of an unstiffened conical shell, the displacements of Eqs. (46) are used also here.

## 8. EVALUATION OF $\beta$ AND $\gamma$ BY COMPLIANCE WITH BOUNDARY CONDITIONS

The shell is assumed to be simply supported. At the end planes ( $x = 1, x_2$ ), the radial displacement  $w$  and the longitudinal moment  $M_x$  must vanish.

A typical term of the displacement series (46) is

$$\begin{aligned} w_n &= \Im C_n x^s \sin t \phi = \Im C_n x^{\gamma+i\beta} \sin t \phi \\ &= \Im C_n x^\gamma [\cos(n\beta \ln x) + i \sin(n\beta \ln x)] \sin t \phi \end{aligned} \quad (48)$$

Hence, its imaginary part is

$$w_n = C_n x^\gamma \sin(n\beta \ln x) \sin t \phi \quad (49)$$

Since  $\ln 1 = 0$ ,  $w$  vanishes identically at the small end plane ( $x = 1$ ) of the cone. In order that  $w$  should also vanish at the large end plane ( $x = x_2$ ),  $\beta$  must be defined as

$$\beta = \pi / \ln x_2 \quad (50)$$

From Eqs. (43) the longitudinal moment

$$M_x = -(D/a) [(1 + \eta_{01}/x) w_{,xx} + \nu (w_{,x}/x + w_{,\phi\phi}/x^2 \sin^2 \alpha) - \zeta_1 u_{,x}/x] \quad (51)$$

At the boundaries considered,  $w_{,\phi\phi}$  is zero and  $u_{,x}$  is very small and may therefore be neglected. Hence, at  $x = 1, x_2$ ,

$$(1 + \eta_{01}/x) w_{,xx} + \nu w_{,x}/x = 0 \quad (52)$$

Substitution of Eq. (49) into Eq. (52) yields for the typical displacement term

$$\begin{aligned} x^{\gamma-2} \sin(n\beta \ln x) \{ [\gamma(\gamma-1) - n^2 \beta^2] (1 + \eta_{01}/x) + \nu \gamma \} \\ + x^{\gamma-2} \cos(n\beta \ln x) n\beta [(2\gamma-1)(1 + \eta_{01}/x) + \nu] = 0 \end{aligned} \quad (53)$$

and after substitution of the boundary conditions

$$(2\gamma-1)(1 + \eta_{01}/x) + \nu = 0 \quad (54)$$

Simultaneous fulfilment of condition (54) at both boundaries is impossible. It can be fulfilled only in one of the boundaries, say at  $x = 1$ , where Eq. (54) becomes

$$(2\gamma-1)(1 + \eta_{01}) + \nu = 0 \quad (55)$$

Hence

$$\gamma = (1/2) [1 - \nu/(1 + \eta_{01})] \quad (56)$$

Equation (56) fulfils the boundary condition  $M_x = 0$  only approximately for stiffened shells. In shells stiffened by stringers, small internal longitudinal moments act at the boundaries. However, in shells stiffened only by frames, the equilibrium boundary conditions of zero longitudinal moments is satisfied rigorously. Since in the absence of stringers

$$\begin{aligned}\eta_{01} &= 0 \\ \zeta_1 &= 0\end{aligned}\tag{57}$$

and Eq. (51) yields, therefore,

$$w_{,xx} + \nu w_{,x} / x = 0\tag{58}$$

By substitution of Eq. (49) into Eq. (58) one obtains then for both  $x = 1$  and  $x = x_2$

$$(2\gamma - 1) + \nu = 0\tag{59}$$

and hence

$$\gamma = (1 - \nu)/2\tag{60}$$

The same result could be obtained by substitution of  $\eta_{01} = 0$  into equation (56). The value of  $\gamma$  for conical shells stiffened by frames only, is the same as that for unstiffened conical shells (Ref. 1), because the frames do not appear in the expression of the longitudinal moment. Hence Eq. (60) fulfils the equilibrium boundary condition of zero longitudinal moments for unstiffened and frame-stiffened conical shells.

## 9. MODIFICATION OF THE FIRST TWO STABILITY EQUATIONS

At the boundaries  $w$  and  $\delta w$  vanish. Hence also the boundary term multiplied by  $\delta w$  in Eq. (45) is zero. Substitution of Eqs. (42) and (43) into the first two stability equations of Eq. (45) yields:

$$\begin{aligned}
 \delta U = & - \int_0^{2\pi} \int_1^x \left\{ [Eh/a(1-\nu^2)] \{ u_{,xx} (1+\mu_1/x) + u_{,x}/x - u(1+\mu_2)/x^2 + (1-\nu) u_{,\phi\phi}/2x^2 \sin^2 \alpha \right. \\
 & + (1+\nu) v_{,x\phi}/2x \sin \alpha - [(3-\nu)/2 + \mu_2] v_{,\phi}/x^2 \sin \alpha - \chi_1 w_{,xxx}/x \\
 & + \chi_2 w_{,x}/x^2 + \chi_2 w_{,\phi\phi}/x^3 \sin^2 \alpha - \nu \cot \alpha w_{,x}/x + (1+\mu_2) \cot \alpha w/x^2 \} \delta u \\
 & + [Eh/a(1-\nu^2)] \{ (1+\nu) u_{,x\phi}/2x \sin \alpha + [(3-\nu)/2 + \mu_2] u_{,\phi}/x^2 \sin \alpha \\
 & + [(1-\nu)/2] v_{,xx} + (1-\nu) v_{,x}/2x + (1+\mu_2) v_{,\phi\phi}/x^2 \sin^2 \alpha \\
 & - (1-\nu) v/2x^2 - \chi_2 w_{,\phi\phi\phi}/x^3 \sin^3 \alpha - \chi_2 w_{,x\phi}/x^2 \sin \alpha \\
 & - (1+\mu_2) \cot \alpha w_{,\phi}/x^2 \sin \alpha \} \delta v + [(xM_x)_{,xx}/a^2 x \\
 & - M_{\phi,x}/a^2 x + M_{\phi,\phi\phi}/a^2 x^2 \sin^2 \alpha - (xM_{x\phi})_{,x\phi}/a^2 x^2 \sin \alpha \\
 & + (xM_{\phi x})_{,x\phi}/a^2 x^2 \sin \alpha + \cot \alpha N_{\phi}/ax - p \tan \alpha (x w_{,xx}/2 + w_{,x} \\
 & + w_{,\phi\phi}/x \sin^2 \alpha) \} \delta w \Big\} a^3 x \sin \alpha dx d\phi \\
 & + \int_0^{2\pi} \{ ax [N_x a \delta u + N_{x\phi} a \delta v - M_x \delta (w_{,x})] \Big\}_{x=1}^{x=x_2} \sin \alpha d\phi = 0 \quad (61)
 \end{aligned}$$

The displacements of Eqs. (46) do not solve the first two stability equations exactly. In order to "correct" this, a set of terms, whose sum is zero, is added to Eq. (61). Then

$$\delta U = - \int_0^{2\pi} \int_1^x \left\{ [Eh/a(1-\nu^2)] \{ (1+\mu_1/x) u_{,xx} + u_{,x}/x - (1+\mu_2) u/x^2 + (1-\nu) u_{,\phi\phi}/2x^2 \sin^2 \alpha \right.$$

$$\begin{aligned}
& + (1+\nu) v_{,x\phi}/2x \sin a - [(3-\nu)/2 + \mu_2] v_{,\phi}/x^2 \sin a - \chi_1 w_{,xxx}/x + \chi_2 w_{,x}/x^2 \\
& + \chi_2 w_{,\phi\phi}/x^3 \sin^2 a - \nu \cot a w_{,x}/x + (1+\mu_2) \cot a w/x^2 \} \delta u \\
& + [Eh/a(1-\nu^2)] [\mu_1 u_{,xx}/k_1 - \mu_1 u_{,xx}/k_1 - \chi_1 x w_{,xxx}/k_2^2 + \chi_1 x w_{,xxx}/k_2^2 \\
& + \chi_2 w_{,x}/x k_3 - \chi_2 w_{,x}/x k_3 + \chi_2 w_{,\phi\phi}/k_4 \sin^2 a x^2 - \chi_2 w_{,\phi\phi}/k_4 \sin^2 a x^2] \delta u \\
& + [Eh/a(1-\nu^2)] \{ (1+\nu) u_{,x\phi}/2x \sin a + [(3-\nu)/2 + \mu_2] u_{,\phi}/x^2 \sin a + (1-\nu) v_{,xx}/2 \\
& + (1-\nu) v_{,x}/2x + (1+\mu_2) v_{,\phi\phi}/x^2 \sin^2 a - (1-\nu) v/2x^2 - \chi_2 w_{,\phi\phi\phi}/x^3 \sin^3 a \\
& - \chi_2 w_{,x\phi}/x^2 \sin a - (1+\mu_2) \cot a w_{,\phi}/x^2 \sin a \} \delta v \\
& + [Eh/a(1-\nu^2)] [-\chi_2 w_{,\phi\phi\phi}/k_5 x^2 \sin^3 a + \chi_2 w_{,\phi\phi\phi}/k_5 x^2 \sin^3 a \\
& - \chi_2 w_{,x\phi}/k_6 x \sin a + \chi_2 w_{,x\phi}/k_6 x \sin a] \delta v \\
& + [(x M_x)_{,xx}/a^2 x - M_{\phi,x}/a^2 x + M_{\phi,\phi\phi}/a^2 x^2 \sin^2 a - (x M_{\phi\phi})_{,x\phi}/a^2 x^2 \sin a \\
& + (x M_{\phi\phi})_{,x\phi}/a^2 x^2 \sin a + \cot a N_{\phi}/ax - p \tan a (x w_{,xx}/2 + w_{,x} \\
& + w_{,\phi\phi}/x \sin^2 a) ] \delta w \} a^3 x \sin a dx d\phi \\
& + \int_0^{2\pi} \int_1^{x_2} \{ ax [N_x a \delta u + N_{x\phi} a \delta v - M_x \delta (w_{,x})] \} \Big|_{x=1}^{x=x_2} \sin a d\phi = 0 \quad (62)
\end{aligned}$$

The expressions multiplied by the second  $\delta u$  and  $\delta v$  are these new terms. After rearrangement of the terms of Eq. (62) one obtains:

$$\begin{aligned}
\delta U = - \int_0^{2\pi} \int_1^{x_2} \{ & [Eh/a(1-\nu^2)] \{ (1+\mu_1/k_1) u_{,xx} + u_{,x}/x - (1+\mu_2) u/x^2 + (1-\nu) u_{,\phi\phi}/2x^2 \sin^2 a \\
& + (1+\nu) v_{,x\phi}/2x \sin a - [(3-\nu)/2 + \mu_2] v_{,\phi}/x^2 \sin a - \chi_1 x w_{,xxx}/k_2^2 + \chi_2 w_{,x}/k_3 x \\
& + \chi_2 w_{,\phi\phi}/k_4 x^2 \sin^2 a - \nu \cot a w_{,x}/x + (1+\mu_2) \cot a w/x^2 \} \delta u \\
& + [Eh/a(1-\nu^2)] \{ (1+\nu) u_{,x\phi}/2x \sin a + [(3-\nu)/2 + \mu_2] u_{,\phi}/x^2 \sin a + (1-\nu) v_{,xx}/2 \\
& + (1-\nu) v_{,x}/2x + (1+\mu_2) v_{,\phi\phi}/x^2 \sin^2 a - (1-\nu) v/2x^2 - \chi_2 w_{,\phi\phi\phi}/x^3 \sin^3 a \\
& - \chi_2 w_{,x\phi}/x^2 \sin a - (1+\mu_2) \cot a w_{,\phi}/x^2 \sin a \} \delta v \\
& + [Eh/a(1-\nu^2)] [-\chi_2 w_{,\phi\phi\phi}/k_5 x^2 \sin^3 a + \chi_2 w_{,\phi\phi\phi}/k_5 x^2 \sin^3 a \\
& - \chi_2 w_{,x\phi}/k_6 x \sin a + \chi_2 w_{,x\phi}/k_6 x \sin a] \delta v \\
& + [(x M_x)_{,xx}/a^2 x - M_{\phi,x}/a^2 x + M_{\phi,\phi\phi}/a^2 x^2 \sin^2 a - (x M_{\phi\phi})_{,x\phi}/a^2 x^2 \sin a \\
& + (x M_{\phi\phi})_{,x\phi}/a^2 x^2 \sin a + \cot a N_{\phi}/ax - p \tan a (x w_{,xx}/2 + w_{,x} \\
& + w_{,\phi\phi}/x \sin^2 a) ] \delta w \} a^3 x \sin a dx d\phi \\
& + \int_0^{2\pi} \int_1^{x_2} \{ ax [N_x a \delta u + N_{x\phi} a \delta v - M_x \delta (w_{,x})] \} \Big|_{x=1}^{x=x_2} \sin a d\phi = 0
\end{aligned}$$

$$\begin{aligned}
& + [Eh/a(1-\nu^2)] [\mu_1 u_{,xx}/x - \mu_1 u_{,xx}/k_1 - \chi_1 w_{,xxx}/x + \chi_1 x w_{,xxx}/k_2^2 \\
& + \chi_2 w_{,x}/x^2 - \chi_2 w_{,x}/k_3 x + \chi_2 w_{,\phi\phi}/x^3 \sin^2 \alpha - \chi_2 w_{,\phi\phi}/k_4 x^2 \sin^2 \alpha] \delta u \\
& + [Eh/a(1-\nu^2)] \{ (1+\nu) u_{,\phi\phi}/2x \sin \alpha + [(3-\nu)/2 + \mu_2] u_{,\phi}/x^2 \sin \alpha + (1-\nu) v_{,xz}/2 \\
& + (1-\nu) v_{,x}/2x + (1+\mu_2) v_{,\phi\phi}/x^2 \sin^2 \alpha - (1-\nu) v/2x^2 - \chi_2 w_{,\phi\phi\phi}/k_5 x^2 \sin^3 \alpha \\
& - \chi_2 w_{,\phi\phi}/k_6 x \sin \alpha - (1+\mu_2) \cot \alpha w_{,\phi}/x^2 \sin \alpha \} \delta v \\
& + [Eh/a(1-\nu^2)] [-\chi_2 w_{,\phi\phi\phi}/x^3 \sin^3 \alpha + \chi_2 w_{,\phi\phi\phi}/k_5 x^2 \sin^3 \alpha \\
& - \chi_2 w_{,\phi\phi}/x^2 \sin \alpha + \chi_2 w_{,\phi\phi}/k_6 x \sin \alpha] \delta v \\
& + [(x M_{xx})_{,xx}/a^2 x - M_{\phi,x}/a^2 x + M_{\phi,\phi\phi}/a^2 x^2 \sin^2 \alpha - (x M_{x\phi})_{,x\phi}/a^2 x^2 \sin \alpha \\
& + (x M_{\phi x})_{,x\phi}/a^2 x^2 \sin \alpha + \cot \alpha N_{\phi}/ax - p \tan \alpha (x w_{,xx}/2 + w_{,x} \\
& + w_{,\phi\phi}/x \sin^2 \alpha)] \delta w \} a^3 x \sin \alpha dx d\phi \\
& + \int_0^{2\pi} \{ a x [N_x a \delta u + N_{x\phi} a \delta v - M_x \delta(w_{,x})] \} \Big|_{x=1}^{x=x_2} \sin \alpha d\phi = 0 \tag{63}
\end{aligned}$$

Now, in Eq. (63) the expression multiplied by the first  $\delta u$  is the first "corrected stability equation" and the expression multiplied by the second  $\delta u$  is the "error-load" of the first stability equation. In the same way, the expression multiplied by the first  $\delta v$  is the second "corrected stability equation" and the expression multiplied by the second  $\delta v$  is its "error-load".  $k_1, k_2, k_3, k_4, k_5$  and  $k_6$  are the "correcting coefficients", which will be calculated by equating the virtual work done by the "error-loads" to zero.

As a matter of fact, for calculation of the "correcting coefficients" it is usually enough to consider only the first term of the displacements series.

## 10. SOLUTION OF THE FIRST TWO "CORRECTED" STABILITY EQUATIONS

The following notation is introduced for brevity:

$$\begin{aligned}
 \mu_1/k_1 &= \mu_{11} \\
 \chi_1/k_2^2 &= \chi_{12} & \chi_2/k_5 &= \chi_{25} \\
 \chi_2/k_3 &= \chi_{23} & \chi_2/k_6 &= \chi_{26} \\
 \chi_2/k_4 &= \chi_{24}
 \end{aligned} \tag{64}$$

Since the displacements of Eqs. (46) solve the first two "corrected" stability equations of Eq.(63) exactly, they may now be written for these displacements as

$$\begin{aligned}
 (1+\mu_{11})u_{,xx} + u_{,x}/x - (1+\mu_2)u/x^2 + (1-\nu)u_{,\phi\phi}/2x^2 \sin^2 a + (1+\nu)v_{,x\phi}/2x \sin a \\
 - [(3-\nu)/2+\mu_2]v_{,\phi}/x^2 \sin a - \chi_{12}xw_{,xxx} + \chi_{23}w_{,x}/x + \chi_{24}w_{,\phi\phi}/x^2 \sin^2 a \\
 - \nu \cot a w_{,x}/x + (1+\mu_2) \cot a w/x^2 = 0 \\
 (1+\nu)u_{,x\phi}/2x \sin a + [(3-\nu)/2+\mu_2]u_{,\phi}/x^2 \sin a + (1-\nu)v_{,xx}/2 + (1-\nu)v_{,x}/2x \\
 + (1+\mu_2)v_{,\phi\phi}/x^2 \sin^2 a - (1-\nu)v/2x^2 - \chi_{25}w_{,\phi\phi\phi}/x^2 \sin^3 a \\
 - \chi_{26}w_{,x\phi}/x \sin a - (1+\mu_2) \cot a w_{,\phi}/x^2 \sin a = 0
 \end{aligned} \tag{65}$$

and after substitution of the displacements of Eqs. (46) into them one obtains

$$\sum_{n=1}^N x^{s-2} \sin t \phi \left\{ A_n [(1+\mu_{11})s(s-1) + s - (1-\nu)t^2/2 \sin^2 a] \right.$$



$$\begin{aligned}
& + B_n \{ -(1+\nu) t s / 2 \sin \alpha + [(3-\nu) / 2 + \mu_2] t / \sin \alpha \} \\
& + C_n [ -\chi_{12} s(s-1)(s-2) + \chi_{23} s - \chi_{24} t^2 / \sin^2 \alpha \\
& - \nu s \cot \alpha + (1 + \mu_2) \cot \alpha ] \} = 0 \\
\sum_{n=1}^N x^{s-2} \cos t \phi & \left\{ A_n \{ (1+\nu) s t / 2 \sin \alpha + [(3-\nu) / 2 + \mu_2] t / \sin \alpha \} \right. \\
& + B_n [ (1-\nu) s(s-1) / 2 + (1-\nu) s / 2 - (1+\mu_2) t^2 / \sin^2 \alpha - (1-\nu) / 2 ] \\
& \left. + C_n [ \chi_{25} t^3 / \sin^3 \alpha - \chi_{26} t s / \sin \alpha - (1+\mu_2) t \cot \alpha / \sin \alpha ] \right\} = 0 \quad (66)
\end{aligned}$$

For Eqs. (66) to be satisfied, every term of their series must be zero. After division by the factors  $x^{s-2} \sin t \phi$  and  $x^{s-2} \cos t \phi$ , which multiply every term of the series in the first or the second equation, respectively, algebraic equations are obtained. The solution of these algebraic equations yields the coefficients  $A_n$  and  $B_n$  as functions of  $C_n$

$$\begin{aligned}
A_n &= (A_{rn} + i A_{in}) C_n \\
B_n &= (B_{rn} + i B_{in}) C_n \quad (67)
\end{aligned}$$

where  $A_{rn}$ ,  $A_{in}$ ,  $B_{rn}$ ,  $B_{in}$  and  $C_n$  are real coefficients defined by:-

$$\begin{aligned}
A_{rn} &= (M_0 M_1 + N_0 N_1) / (M_0^2 + N_0^2) \\
A_{in} &= (M_0 N_1 - N_0 M_1) / (M_0^2 + N_0^2) \\
B_{rn} &= (M_0 M_2 + N_0 N_2) / (M_0^2 + N_0^2) \\
B_{in} &= (M_0 N_2 - N_0 M_2) / (M_0^2 + N_0^2) \quad (68)
\end{aligned}$$

where

$$\begin{aligned}
 M_0(n) &= M_{01} t^4 + M_{02} t^2 + M_{03} \\
 N_0(n) &= N_{01} t^2 + N_{02} \\
 M_1(n) &= M_{11} t^4 + M_{12} t^2 + M_{13} \\
 N_1(n) &= N_{11} t^4 + N_{12} t^2 + N_{13} \\
 M_2(n) &= M_{21} t^5 + M_{22} t^3 + M_{23} t \\
 N_2(n) &= N_{21} t^3 + N_{22} t
 \end{aligned} \tag{69}$$

$$\begin{aligned}
 M_{01}(n) &= d_{01} / \sin^4 \alpha \\
 M_{02}(n) &= (1/\sin^2 \alpha) [n^2 \beta^2 (-d_{02}) + \gamma^2 d_{02} + \gamma d_{03} + d_{04}] \\
 M_{03}(n) &= n^4 \beta^4 d_{05} + n^2 \beta^2 (-6 \gamma^2 d_{05} - 3 \gamma d_{06} - d_{07}) + (\gamma^4 d_{05} + \gamma^3 d_{06} \\
 &\quad + \gamma^2 d_{07} + \gamma d_{08} + d_{09})
 \end{aligned} \tag{70}$$

$$\begin{aligned}
 N_{01}(n) &= (n \beta / \sin^2 \alpha) (2 \gamma d_{02} + d_{03}) \\
 N_{02}(n) &= n^3 \beta^3 (-4 \gamma d_{05} - d_{06}) + n \beta (4 \gamma^3 d_{05} + 3 \gamma^2 d_{06} + 2 \gamma d_{07} + d_{08})
 \end{aligned} \tag{71}$$

$$\begin{aligned}
 M_{11}(n) &= (1/\sin^4 \alpha) (\gamma d_{11} + d_{12}) \\
 M_{12}(n) &= (1/\sin^2 \alpha) [n^2 \beta^2 (-3 \gamma d_{13} - d_{14}) + (\gamma^3 d_{13} + \gamma^2 d_{14} + \gamma d_{15} + d_{16})] \\
 M_{13}(n) &= n^4 \beta^4 (5 \gamma d_{17} + d_{18}) + n^2 \beta^2 (-10 \gamma^3 d_{17} - 6 \gamma^2 d_{18} - 3 \gamma d_{19} - d_{110}) \\
 &\quad + (\gamma^5 d_{17} + \gamma^4 d_{18} + \gamma^3 d_{19} + \gamma^2 d_{110} + \gamma d_{111} + d_{112})
 \end{aligned} \tag{72}$$

$$N_{11}(n) = n \beta d_{11} / \sin^4 a$$

$$N_{12}(n) = (1/\sin^2 a) [n^3 \beta^3 (-d_{13}) + n \beta (3 \gamma^2 d_{13} + 2 \gamma d_{14} + d_{15})]$$

$$N_{13}(n) = n^5 \beta^5 d_{17} + n^3 \beta^3 (-10 \gamma^2 d_{17} - 4 \gamma d_{18} - d_{19}) \\ + n \beta (5 \gamma^4 d_{17} + 4 \gamma^3 d_{18} + 3 \gamma^2 d_{19} + 2 \gamma d_{110} + d_{111}) \quad (73)$$

$$M_{21}(n) = d_{21} / \sin^5 a$$

$$M_{22}(n) = (1/\sin^3 a) [n^2 \beta^2 (-d_{22}) + (\gamma^2 d_{22} + \gamma d_{23} + d_{24})]$$

$$M_{23}(n) = (1/\sin a) [n^4 \beta^4 d_{25} + n^2 \beta^2 (-6 \gamma^2 d_{25} - 3 \gamma d_{26} - d_{27}) \\ + (\gamma^4 d_{25} + \gamma^3 d_{26} + \gamma^2 d_{27} + \gamma d_{28} + d_{29})] \quad (74)$$

$$N_{21}(n) = (n \beta / \sin^3 a) (2 \gamma d_{22} + d_{23})$$

$$N_{22}(n) = (1/\sin a) [n^3 \beta^3 (-4 \gamma d_{25} - d_{26}) + n \beta (4 \gamma^3 d_{25} + 3 \gamma^2 d_{26} \\ + 2 \gamma d_{27} + d_{28})] \quad (75)$$

$$d_{01} = (1 - \nu) (1 + \mu_2)$$

$$d_{02} = 2 \nu - 2(1 + \mu_2) (1 + \mu_{11})$$

$$d_{03} = 2 \mu_{11} (1 + \mu_2)$$

$$d_{04} = -2(1 - \nu) (1 + \mu_2)$$

$$d_{05} = (1 - \nu) (1 + \mu_{11})$$

$$d_{06} = -(1 - \nu) \mu_{11}$$

$$\begin{aligned}
d_{07} &= -(1-\nu)(\mu_{11} + \mu_2 + 2) \\
d_{08} &= (1-\nu)\mu_{11} \\
d_{09} &= (1-\nu)(1 + \mu_2)
\end{aligned} \tag{76}$$

$$\begin{aligned}
d_{11} &= -(1+\nu)\chi_{25} \\
d_{12} &= -(1+\mu_2)^2\chi_{24} + (3-\nu+2\mu_2)\chi_{25} \\
d_{13} &= -2(1+\mu_2)\chi_{12} \\
d_{14} &= 6(1+\mu_2)\chi_{12} + (1-\nu)\chi_{24} + (1+\nu)\chi_{26} \\
d_{15} &= -4(1+\mu_2)\chi_{12} + 2(1+\mu_2)\chi_{23} - (3-\nu+2\mu_2)\chi_{26} + (1-\nu)(1+\mu_2)\cot\alpha \\
d_{16} &= -(1-\nu)\chi_{24} - (1-\nu)(1+\mu_2)\cot\alpha \\
d_{17} &= (1-\nu)\chi_{12} \\
d_{18} &= -3(1-\nu)\chi_{12} \\
d_{19} &= (1-\nu)\chi_{12} - (1-\nu)\chi_{23} + \nu(1-\nu)\cot\alpha \\
d_{110} &= 3(1-\nu)\chi_{12} - (1-\nu)(1+\mu_2)\cot\alpha \\
d_{111} &= -2(1-\nu)\chi_{12} + (1-\nu)\chi_{23} - \nu(1-\nu)\cot\alpha \\
d_{112} &= (1-\nu)(1+\mu_2)\cot\alpha
\end{aligned} \tag{77}$$

$$\begin{aligned}
d_{21} &= (1-\nu)\chi_{25} \\
d_{22} &= -2(1+\mu_{11})\chi_{25}
\end{aligned}$$

$$\begin{aligned}
d_{23} &= 2 \mu_{11} \chi_{25} - (1 + \nu) \chi_{24} - (1 - \nu) \chi_{26} \\
d_{24} &= -(3 - \nu + 2 \mu_2) \chi_{24} + 2(1 + \mu_2) \chi_{25} + (1 - \nu)(1 + \mu_2) \cot a \\
d_{25} &= -(1 + \nu) \chi_{12} \\
d_{26} &= 2(2\nu - \mu_2) \chi_{12} + 2 \chi_{26} (1 + \mu_{11}) \\
d_{27} &= -2 \mu_{11} \chi_{26} + (7 - 5\nu + 6 \mu_2) \chi_{12} + (1 + \nu) \chi_{23} \\
&\quad + [2(1 + \mu_2)(1 + \mu_{11}) - \nu(1 + \nu)] \cot a \\
d_{28} &= -2 \mu_{11} (1 + \mu_2) \cot a - 2(3 - \nu + 2 \mu_2) \chi_{12} + (3 - \nu + 2 \mu_2) \chi_{23} \\
&\quad - 2(1 + \mu_2) \chi_{26} + (1 - \nu)(1 - \nu + \mu_2) \cot a \\
d_{29} &= (1 - \nu)(1 + \mu_2) \cot a
\end{aligned} \tag{78}$$

Now, since

$$x^{\alpha} = x^{\gamma + i n \beta} = x^{\gamma} [\cos(n \beta \ln x) + i \sin(n \beta \ln x)] \tag{79}$$

the displacements become, after substitution of Eqs. (67) into Eqs. (46)

$$\begin{aligned}
u &= \sum_{n=1}^N C_n (A_{rn} + i A_{in}) x^{\gamma} [\cos(n \beta \ln x) + i \sin(n \beta \ln x)] \sin t \phi \\
v &= \sum_{n=1}^N C_n (B_{rn} + i B_{in}) x^{\gamma} [\cos(n \beta \ln x) + i \sin(n \beta \ln x)] \cos t \phi \\
w &= \sum_{n=1}^N C_n x^{\gamma} [\cos(n \beta \ln x) + i \sin(n \beta \ln x)] \sin t \phi
\end{aligned} \tag{80}$$

The imaginary part of these equations is

$$\begin{aligned}
u &= \sin t\phi \sum_{n=1}^N x^\gamma [A_{rn} \sin(n\beta \ln x) + A_{in} \cos(n\beta \ln x)] C_n \\
v &= \cos t\phi \sum_{n=1}^N x^\gamma [B_{rn} \sin(n\beta \ln x) + B_{in} \cos(n\beta \ln x)] C_n \\
w &= \sin t\phi \sum_{n=1}^N x^\gamma [\sin(n\beta \ln x)] C_n
\end{aligned} \tag{81}$$

where all the values are real.

The three displacements are thus expressed as functions of one set of real arbitrary coefficients  $C_n$ . Hence the variations of the displacements are

$$\begin{aligned}
\delta u &= \sin t\phi \sum_{n=1}^N x^\gamma [A_{rn} \sin(n\beta \ln x) + A_{in} \cos(n\beta \ln x)] \delta C_n \\
\delta v &= \cos t\phi \sum_{n=1}^N x^\gamma [B_{rn} \sin(n\beta \ln x) + B_{in} \cos(n\beta \ln x)] \delta C_n \\
\delta w &= \sin t\phi \sum_{n=1}^N x^\gamma [\sin(n\beta \ln x)] \delta C_n
\end{aligned} \tag{82}$$

## 11. CORRECTING COEFFICIENTS

In Section 9 "correcting coefficients" were introduced which then, in Section 10, permitted the expression of the  $u$  and  $v$  displacements as functions of the radial one,  $w$ . These "correcting coefficients" will now be evaluated by equating the virtual work done by the error-loads of the first two stability equations of Eq. (63), to zero.

$$\begin{aligned}
0 &= - \int_0^{2\pi} \int_1^2 \{ [Fh/a (1-\nu^2)] [u_{,xx} \mu_1/x - u_{,xx} \mu_1/k_1 - \chi_1 w_{,xxx}/x + \chi_1 x w_{,xxx}/k_2^2 \\
&\quad + \chi_2 w_{,x}/x^2 - \chi_2 w_{,x}/k_3 x + \chi_2 w_{,\phi\phi}/x^3 \sin^2 \alpha - \chi_2 w_{,\phi\phi}/k_4 x^2 \sin \alpha] \delta u
\end{aligned}$$

$$\begin{aligned}
& + [Eh/a (1-\nu^2)] [-\chi_2 w_{,\phi\phi\phi}/x^3 \sin^3 \alpha + \chi_2 w_{,\phi\phi\phi}/k_5 x^2 \sin^3 \alpha \\
& - \chi_2 w_{,x\phi}/x^2 \sin \alpha + \chi_2 w_{,x\phi}/k_6 x \sin \alpha] \delta v \int a^3 x \sin \alpha dx d\phi
\end{aligned} \quad (83)$$

If all corresponding pairs are equal to zero, the entire Eq. (83) vanishes too. The "correcting coefficients" may be calculated in this manner. For example, for  $k_1$  one obtains:-

$$-\int_0^{2\pi} \int_1^{x_2} [Eh/a (1-\nu^2)] [u_{,xx} \mu_1/x - u_{,xx} \mu_1/k_1] u a^3 x \sin \alpha dx d\phi = 0 \quad (84)$$

and then

$$k_1 = \frac{\int_0^{2\pi} \int_1^{x_2} u_{,xx} u x dx d\phi}{\int_0^{2\pi} \int_1^{x_2} u_{,xx} u dx d\phi} \quad (85)$$

Similarly the remaining "correcting coefficients" are obtained:-

$$k_2 = \frac{\int_0^{2\pi} \int_1^{x_2} w_{,xxx} u x^2 dx d\phi}{\int_0^{2\pi} \int_1^{x_2} w_{,xxx} u dx d\phi}$$

$$k_3 = \frac{\int_0^{2\pi} \int_1^{x_2} w_{,x} u dx d\phi}{\int_0^{2\pi} \int_1^{x_2} (w_{,x}/x) u dx d\phi}$$

$$k_4 = \frac{\int_0^{2\pi} \int_1^{x_2} (w, \phi \phi / x) u \, dx \, d\phi}{\int_0^{2\pi} \int_1^{x_2} (w, \phi \phi / x^2) \, dx \, d\phi}$$

$$k_5 = \frac{\int_0^{2\pi} \int_1^{x_2} (w, \phi \phi \phi / x) v \, dx \, d\phi}{\int_0^{2\pi} \int_1^{x_2} (w, \phi \phi \phi / x^2) v \, dx \, d\phi}$$

$$k_6 = \frac{\int_0^{2\pi} \int_1^{x_2} w, x \phi \, v \, dx \, d\phi}{\int_0^{2\pi} \int_1^{x_2} (w, x \phi / x) v \, dx \, d\phi}$$

(86)

Substitution of the displacements, Eqs. (81), into Eqs. (85) and (86) yields:

$$k_1 = \frac{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-1}(n,m) A_{cn} A_{rm} + I_2^{2\gamma-1}(m,n) A_{cn} A_{im} + I_2^{2\gamma-1}(n,m) A_{dn} A_{rm} + I_3^{2\gamma-1}(n,m) A_{dn} A_{im}]}{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-2}(n,m) A_{cn} A_{rm} + I_2^{2\gamma-2}(m,n) A_{cn} A_{im} + I_2^{2\gamma-2}(n,m) A_{dn} A_{rm} + I_3^{2\gamma-2}(n,m) A_{dn} A_{im}]} \quad (87)$$

$$k_2 = \frac{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-1}(n,m) a_{3n} A_{rm} + I_2^{2\gamma-1}(m,n) a_{3n} A_{im} + I_2^{2\gamma-1}(n,m) b_{3n} A_{rm} + I_3^{2\gamma-1}(n,m) b_{3n} A_{im}]}{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-3}(n,m) a_{3n} A_{rm} + I_2^{2\gamma-3}(m,n) a_{3n} A_{im} + I_2^{2\gamma-3}(n,m) b_{3n} A_{rm} + I_3^{2\gamma-3}(n,m) b_{3n} A_{im}]} \quad (88)$$



$$k_3 = \frac{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-1}(n,m) \gamma A_{rm} + I_2^{2\gamma-1}(m,n) \gamma A_{im} + I_2^{2\gamma-1}(n,m) n \beta A_{rm} + I_3^{2\gamma-1}(n,m) n \beta A_{im}]}{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-2}(n,m) \gamma A_{rm} + I_2^{2\gamma-2}(m,n) \gamma A_{im} + I_2^{2\gamma-2}(n,m) n \beta A_{rm} + I_3^{2\gamma-2}(n,m) n \beta A_{im}]} \quad (89)$$

$$k_4 = \frac{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-1}(n,m) A_{rm} + I_2^{2\gamma-1}(m,n) A_{im}]}{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-2}(n,m) A_{rm} + I_2^{2\gamma-2}(m,n) A_{im}]} \quad (90)$$

$$k_5 = \frac{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-1}(n,m) B_{rm} + I_2^{2\gamma-1}(m,n) B_{im}]}{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-2}(n,m) B_{rm} + I_2^{2\gamma-2}(m,n) B_{im}]} = \frac{\kappa_5^{2\gamma-1}}{\kappa_5^{2\gamma-2}} \quad (91)$$

$$k_6 = \frac{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-1}(n,m) \gamma B_{rm} + I_2^{2\gamma-1}(m,n) \gamma B_{im} + I_2^{2\gamma-1}(n,m) n \beta B_{rm} + I_3^{2\gamma-1}(n,m) n \beta B_{im}]}{\sum_{n=1}^N \sum_{m=1}^N C_n C_m [I_1^{2\gamma-2}(n,m) \gamma B_{rm} + I_2^{2\gamma-2}(m,n) \gamma B_{im} + I_2^{2\gamma-2}(n,m) n \beta B_{rm} + I_3^{2\gamma-2}(n,m) n \beta B_{im}]} = \frac{\kappa_6^{2\gamma-1}}{\kappa_6^{2\gamma-2}} \quad (92)$$

It is possible to combine several correcting coefficients in order to reduce the number of coefficients. For example, the coefficient  $k_6$  was found, during the calculations, to be very small. Hence, it was combined with  $k_5$  and redefined as

$$k_5 = k_6 = k_{56} \quad (93)$$

From Eq. (83), one obtains then

$$k_{56} = \frac{\int_0^{2\pi} \int_1^{x_2} (w_{,\phi\phi\phi}/x^2 \sin^3 \alpha + w_{,x\phi}/x \sin \alpha) v x dx d\phi}{\int_0^{2\pi} \int_1^{x_2} (w_{,\phi\phi\phi}/x^3 \sin^3 \alpha + w_{,x\phi}/x^2 \sin \alpha) v x dx d\phi} \quad (94)$$

and finally

$$k_{56} = \frac{(t^2/\sin^2 \alpha) k_5^{2\gamma-1} - k_6^{2\gamma-1}}{(t^2/\sin^2 \alpha) k_5^{2\gamma-2} - k_6^{2\gamma-2}} \quad (95)$$

The various terms which appear in Eqs. (86) to (94) are defined as follows:-

$$\begin{aligned} A_{cn} &= [\gamma(\gamma-1) - n^2 \beta^2] A_{rn} - n\beta(2\gamma-1) A_{in} \\ A_{dn} &= [\gamma(\gamma-1) - n^2 \beta^2] A_{in} + n\beta(2\gamma-1) A_{rn} \end{aligned} \quad (96)$$

$$a_{2n} = \gamma(\gamma-1) - n^2 \beta^2$$

$$b_{2n} = n\beta(2\gamma-1)$$

$$a_{3n} = \gamma(\gamma-1)(\gamma-2) - n^2 \beta^2(3\gamma-3)$$

$$b_{3n} = n\beta(3\gamma^2 + 6\gamma + 2 - n^2 \beta^2) \quad (97)$$

$$\begin{aligned} I_1^K(n, m) &= \int_1^{x_2} x^K \sin(n\beta \ln x) \sin(m\beta \ln x) dx \\ &= (1/2)(\kappa+1) [1 - x_2^{\kappa+1} (-1)^{n+m}] \left[ \frac{1}{(n+m)^2 \beta^2 + (\kappa+1)^2} - \frac{1}{(n-m)^2 \beta^2 + (\kappa+1)^2} \right] \end{aligned}$$

$$\begin{aligned}
I_2^\kappa(n, m) &= \int_1^{x_2} x^\kappa \cos(n \beta \ln x) \sin(m \beta \ln x) dx \\
&= (1/2) \beta [1 - x_2^{\kappa+1} (-1)^{n+m}] \left[ \frac{n+m}{(n+m)^2 \beta^2 + (\kappa+1)^2} + \frac{m-n}{(m-n)^2 \beta^2 + (\kappa+1)^2} \right] \\
I_3^\kappa(n, m) &= \int_1^{x_2} x^\kappa \cos(n \beta \ln x) \cos(m \beta \ln x) dx \\
&= (1/2) (\kappa+1) [x_2^{\kappa+1} (-1)^{n+m} - 1] \left[ \frac{1}{(n+m)^2 \beta^2 + (\kappa+1)^2} + \frac{1}{(n-m)^2 \beta^2 + (\kappa+1)^2} \right] \quad (98)
\end{aligned}$$

The physical interpretation of the "correcting coefficients" is that they introduce artificial ties between the displacements  $u$ ,  $v$  and  $w$ , which reduce the degrees of freedom of the shell. The shell is thus artificially stiffened and the critical load obtained is higher than the actual one. For calculation of the correcting coefficients, the first term of the displacement series, which represents the dominant component of the buckling mode under hydrostatic pressure, is usually sufficient. If the first approximation yields the required accuracy, the correcting coefficients have no effect on the critical load. Since addition of the second, third and higher terms of the displacement series only improves the one-term solution, the increase of the critical load due to the correcting coefficients must be negligible.

It should be pointed out that the coefficients  $A_{rn}$ ,  $A_{in}$ ,  $B_{rn}$ , and  $B_{in}$  which appear in the right side of Eqs. (87) to (92) are themselves functions of the correcting coefficients. Hence an iteration procedure is required. One first assumes values for the correcting coefficients, and after calculation of the critical load checks if they were assumed properly. If large differences are obtained, the critical load has to be recalculated with the new correcting coefficients. Since the requirement of zero virtual work done by the error-loads of the two stability equations is not a mandatory requirement, but is only a means for improvement of the accuracy of the solution, the correcting coefficients need not be determined very accurately. The correcting coefficients can, therefore, be calculated with the coefficients  $A_{rn}$ ,  $A_{in}$ ,  $B_{rn}$  and  $B_{in}$  obtained from the approximate solution given in Section 15, where the latter coefficients are not functions of the former.

Although the use of the correcting coefficients introduces artificial ties between the displacements

$u$ ,  $v$  and  $w$ , these ties are not far from the actual ones. In Ref. 13, the authors have shown that the rigorous solution of the Donnell type stability equations for an unstiffened cylindrical shell under hydrostatic pressure is valid also for those of a stiffened cylindrical shell failing by general instability, since the behaviour of a stiffened cylindrical shell is similar to that of an unstiffened one. It is reasonable to assume that the same will occur for conical shells. Hence one may expect that the buckling displacements of unstiffened conical shells will be suitable for the case of general instability of stiffened conical shells. However, though these displacements solved the first two stability equations exactly in the case of unstiffened conical shells, they do so in the case of stiffened conical shells only after the first two stability equations have been "corrected" slightly by the correcting coefficients.

## 12. BOUNDARY CONDITIONS

The buckling displacements have to satisfy the following boundary conditions :

### a. Geometrical boundary conditions

$$\begin{aligned} w &= 0 \\ &\text{at } x = 1 \text{ and } x = x_2 \\ v &= 0 \end{aligned} \tag{99}$$

### b. Equilibrium boundary conditions

$$\begin{aligned} N_x &= 0 \\ &\text{at } x = 1 \text{ and } x = x_2 \\ M_x &= 0 \end{aligned} \tag{100}$$

The displacements of Eqs. (46) do not fulfil all the boundary conditions. In Section 8 the values of  $\gamma$  and  $\beta$  were determined from the compliance with the boundary condition of zero radial displacement, and also of zero longitudinal moment, when the shell is stiffened by rings only. The displacements do not fulfil the boundary conditions of zero circumferential displacement and of zero longitudinal normal force. In case of stiffening by stringers, the longitudinal moment in the boundaries is also not zero.

Hence in spite of the requirements of Eqs. (99) and (100), the displacements yield

$$\begin{aligned} v &\neq 0 \\ N_x &\neq 0 \quad \text{at } x = 1 \text{ and } x = x_2 \\ M_x &\neq 0 \end{aligned} \quad (101)$$

In order to evaluate the effect of this non-compliance with boundary conditions upon the critical load, modified boundary conditions were proposed in Ref. 1. The ends of the shell were assumed to be elastically restrained. For a stiffened conical shell, with such modified boundary conditions, the forces and the moments appearing in the elastic supports are equal to the internal forces and moments acting at the boundaries. The restraining forces are obtained from Eqs. (43). At  $x = 1$  and  $x = x_2$

$$N_x = [Eh/(1-\nu^2)] [(1 + \mu_1/x)u_{,x} + \nu(v_{,\phi}/x \sin \alpha + u/x - w \cot \alpha/x) - \chi_1 w_{,xx}/x] \quad (102)$$

$$N_{x\phi} = [Eh/2(1+\nu)] [v_{,x} - v/x + u_{,\phi}/x \sin \alpha] \quad (103)$$

$$M_x = -(D/a)[(1 + \eta_{01}/x)w_{,xx} + \nu(w_{,x}/x + w_{,\phi\phi}/x^2 \sin^2 \alpha) - \zeta_1 u_{,x}/x] \quad (104)$$

The spring constants of the elastic supports are defined as

$$\begin{aligned} k_u &= N_x / h a u \quad (\text{psi per inch}) \\ k_v &= N_{x\phi} / h a v \quad (\text{psi per inch}) \quad \text{at } x = 1 \text{ and } x = x_2 \\ k_m &= 6 M_x / h^3 w_{,x} \quad (\text{psi per inch}) \end{aligned} \quad (105)$$

Hence for  $x = 1$ , and  $x = x_2$

$$\frac{k_u}{E} = \frac{1}{(1-\nu^2)ax} \frac{\sum_{n=1}^N C_n \cos(n\beta \ln x) [\gamma A_{in} + n\beta A_{rn} (1 + \mu_1/x) - \nu(t B_{in}/\sin \alpha - A_{in}) - \chi_1 x^{-2n\beta(2\gamma-1)}]}{\sum_{n=1}^N C_n \cos(n\beta \ln x) B_{in}}$$

$$\begin{aligned}
\frac{k_v}{E} &= \frac{1}{2(1+\nu)ax} \frac{\sum_{n=1}^N C_n \cos(n\beta \ln x) [B_{in}(\gamma-1) + n\beta B_{rn} + t A_{in}/\sin \alpha]}{\sum_{n=1}^N C_n \cos(n\beta \ln x) B_{in}} \\
\frac{k_m}{E} &= \frac{1}{2(1-\nu^2)ax} \frac{\sum_{n=1}^N C_n \cos(n\beta \ln x) [n\beta(2\gamma-1)(1+\eta_{01}/x) + \nu n\beta - \zeta_1(\gamma A_{in} + n\beta A_{rn})]}{\sum_{n=1}^N C_n \cos(n\beta \ln x) n\beta}
\end{aligned} \quad (106)$$

In order to compare the magnitude of the spring constants for stiffened shells with those for unstiffened ones the following cases are considered:

T A B L E I.

$\alpha = 30^\circ$	$a = 57.59''$		$h = 0.1''$		$x_2 = 1.5$	
Shell Type	$A_1/b_0 h$	$e_1/h$	$12I_{11}/b_0 h^3$	$A_2/a_0 h$	$e_2/h$	$12I_{22}/a_0 h^3$
a. unstiffened	—	—	—	—	—	—
b. stiffened by internal frames	—	—	—	0.1471	+ 1.653	0.7819
c. stiffened by external frames	—	—	—	0.1471	- 1.653	0.7819
d. stiffened by frames (with their eccentricity neglected)	—	—	—	0.1471	$\pm 1.653$	0.7819
e. stiffened by internal stringers	0.1471	+ 1.653	0.7819	—	—	—
f. stiffened by external stringers	0.1471	- 1.653	0.7819	—	—	—

Taking  $\nu = 0.3$  and  $n = 1$ , one obtains at  $x = 1$  (where the largest values occur):

T A B L E I I.

Shell	a	b	c	d	e	f
$K_u/E$	- 0.0059	- 0.0056	- 0.0071	- 0.0061	- 0.0056	- 0.0048
$K_v/E$	- 1.43	- 2.16	- 1.13	- 1.57	- 1.42	- 1.71
$K_m/E$	-	-	-	-	- 0.0052	+ 0.0027

The comparison shows that the magnitudes of the spring constants of unstiffened and stiffened conical shells are of the same order. Since the values of  $k_v$  are very large, the solution approaches the realistic boundary condition of zero circumferential displacement ( $v = 0$ ). The influence of  $k_u$  upon the critical load was shown in Ref. 2 to be negligible.

Another method for evaluation of the effect of the non-compliance with the boundary conditions and the resulting reduction of the critical load is proposed here.

In the expression of the virtual work during buckling, Eq. (61), the following integral appears.

$$+ \int_0^{2\pi} \left\{ N_x [N_x a \delta u + N_x \phi a \delta v - M_x \delta(w_{,x})] \right\}_{x=1}^{x=x_2} \sin \alpha d\phi \quad (107)$$

This is the virtual work done by the internal forces and moments appearing at the boundaries, or the "boundary work". Since the internal forces and moments at the boundaries are equal to the forces and moments acting upon the assumed boundary springs, the "boundary work" is equal to the work done by these fictitious springs. Now by calculation of the critical load with or without the virtual work done by the "springs", the effect of every "spring" separately, or of all the "springs" together, can be evaluated. For typical shells, stiffened by stringers, the effect of  $M_x (k_m)$  was found to be less than 0.2%. This is not surprising since as a result of the definition of  $\gamma$ , given in Eq. (56), the longitudinal moment  $M_x$  at the boundaries is nearly zero. The effect of  $N_x (k_u)$  upon the critical pressure  $p_{cr}$  is

the most pronounced, being 1% of  $p_{cr}$  for short shells, and 3% – 4% for long ones. It is interesting to note that the effect of  $N_x (k_v)$  is indeed small, as was assumed in Ref. 1, being less than 1% for both short and long shells.

It has been shown in Ref. 14 that by the method of virtual displacements the equilibrium boundary conditions are fulfilled for the complete displacements, although every term of the infinite displacement series does not fulfil them. Hence, the magnitude of  $M_x$  and  $N_x$  at the boundaries, for every term, can only affect the rate of convergence, and not the solution itself, if their "boundary work" is taken into account.

The method of virtual displacements requires that the displacements fulfil the geometrical boundary conditions. The non-compliance of the condition of zero circumferential displacement at the supported ends has therefore to be looked into. It should be noted that although  $v$  does not vanish at the boundaries, it is nearly zero there. In order to show this, the circumferential displacement of a typical shell, shell (b), has been computed. Minimum  $p_{cr}$  appears when the number of circumferential waves  $t$ , is 10. Then

$$\begin{aligned} v_1 &= \cos(10\phi) x^{0.35} [-0.0818744 \sin(\beta \ln x) + 0.00065850 \cos(\beta \ln x)] C_1 \\ v_2 &= \cos(10\phi) x^{0.35} [-0.07556033 \sin(2\beta \ln x) + 0.00010874 \cos(2\beta \ln x)] C_2 \end{aligned} \quad (108)$$

From the above equations it can be seen that  $v_{max}$  at the boundaries is only about 1/100 of  $v_{max}$  appearing in the shell. Nevertheless, the exact effect of the displacement  $v$  at the boundaries upon the critical load can be estimated only by consideration of the "boundary work" of  $N_x$ . Such an analysis shows that the effect is less than 1%.

It may be concluded, therefore, that the boundary conditions applied here for circular conical shells stiffened in two directions, differ only slightly from the usual simple support conditions, and Eqs. (99) and (100) are satisfied with sufficient accuracy.



### 13. COMPLIANCE WITH BOUNDARY CONDITION OF ZERO CIRCUMFERENTIAL DISPLACEMENT

The effect of approximate compliance only with the boundary condition  $v = 0$  on the critical pressure is very small. However, if required (for example for some other type of load), it is possible to fulfil this boundary condition for the complete displacement although not for every term of its series.

Then, though

$$v_n \neq 0 \quad \text{at} \quad x = 1 \quad \text{and} \quad x = x_2 \quad (109)$$

one can prescribe

$$v = \sum_{n=1}^N v_n = 0 \quad \text{at} \quad x = 1 \quad \text{and} \quad x = x_2 \quad (110)$$

From Eqs. (81) the displacement  $v_n$  at the boundaries is:

$$v_n = \cos t \phi \, x^\gamma B_{in} \cos(n \beta \ln x) C_n \quad \text{at} \quad x = 1 \quad \text{and} \quad x = x_2 \quad (111)$$

If

$$1^\gamma B_{in} \cos(n \beta \ln 1) = B_{in}$$

$$x_2^\gamma B_{in} \cos(n \beta \ln x_2) = B_{jn} \quad (112)$$

one obtains

$$v_n = \cos t \phi \, B_{in} C_n \quad \text{at} \quad x = 1$$

$$v_n = \cos t \phi \, B_{jn} C_n \quad \text{at} \quad x = x_2 \quad (113)$$

and the boundary condition, Eq. (110) becomes

$$\sum_{n=1}^N B_{in} C_n = 0$$

$$\sum_{n=1}^N B_{jn} C_n = 0 \quad (114)$$

Eqs. (114) represent the necessary connections between the coefficients  $C_n$  for the compliance with the boundary condition of zero circumferential displacement. Hence, now not all the coefficients  $C_n$  are arbitrary. It may be assumed that the last two coefficients  $C_{N-1}$  and  $C_N$  are functions of the other  $N-2$  arbitrary ones. The solution of Eqs. (114) yields then

$$C_{N-1} = \sum_{n=1}^{N-2} g_{in} C_n \quad \text{where} \quad g_{in} = g_{in}(B_{in}, B_{jn})$$

and

$$C_N = \sum_{n=1}^{N-2} g_{jn} C_n \quad \text{where} \quad g_{jn} = g_{jn}(B_{in}, B_{jn}) \quad (115)$$

and the radial displacement can be written as

$$w = \sin t \phi \left\{ \sum_{n=1}^{N-2} x^y \sin(n \beta \ln x) C_n + x^y \sin[(N-1)\beta \ln x] \sum_{n=1}^{N-2} g_{in} C_n + x^y \sin(N \beta \ln x) \sum_{n=1}^{N-2} g_{jn} C_n \right\}$$

Since one can also obtain  $u$  and  $v$  in a similar manner, the displacements become

$$\begin{aligned} u = \sin t \phi \sum_{n=1}^{N-2} x^y & \left\{ [A_{rn} \sin(n \beta \ln x) + A_{in} \cos(n \beta \ln x)] \right. \\ & + g_{in} [A_{r(N-1)} \sin[(N-1)\beta \ln x] + A_{i(N-1)} \cos[(N-1)\beta \ln x]] \\ & \left. + g_{jn} [A_{rN} \sin(N \beta \ln x) + A_{iN} \cos(N \beta \ln x)] \right\} C_n \\ v = \cos t \phi \sum_{n=1}^{N-2} x^y & \left\{ [B_{rn} \sin(n \beta \ln x) + B_{in} \cos(n \beta \ln x)] \right. \\ & + g_{in} [B_{r(N-1)} \sin[(N-1)\beta \ln x] + B_{i(N-1)} \cos[(N-1)\beta \ln x]] \\ & \left. + g_{jn} [B_{rN} \sin(N \beta \ln x) + B_{iN} \cos(N \beta \ln x)] \right\} C_n \end{aligned}$$

$$w = \sin t \phi \sum_{n=1}^{N-2} x^\gamma \{ \sin(n \beta \ln x) + g_{in} \sin[(N-1)\beta \ln x] + g_{jn} \sin(N \beta \ln x) \} C_n \quad (116)$$

Where all the  $N-2$  coefficients  $C_n$  are arbitrary.

The displacements given by Eqs. (116) fulfil all the geometrical boundary conditions of simple supports. Every term of the displacements fulfils the boundary condition of zero displacement in the radial direction ( $w = 0$ ), and although not every term fulfils the boundary condition of zero circumferential displacement ( $v = 0$ ) the whole series does.

By substitution of Eqs. (43) and (116) into Eq. (63), and integration (with the "boundary work" included) one obtains, since the first  $N-2$  coefficients  $C_n$  are arbitrary,

$$\begin{aligned} \sum_{n=1}^{N-2} C_n [ & T(n, m) + g_{in} T(N-1, m) + g_{jn} T(N, m) \\ & + g_{im} T(n, N-1) + g_{in} g_{im} T(N-1, N-1) + g_{jn} g_{im} T(N, N-1) \\ & + g_{jm} T(n, N) + g_{in} g_{jm} T(N-1, N) + g_{jn} g_{jm} T(N, N)] = 0 \end{aligned} \quad (m = 1 \text{ to } N-2) \quad (117)$$

where  $T(n, m)$  is the virtual work, multiplied by some known coefficient,  $n$  denotes the stress state and  $m$  the state of deformation. The lowest eigenvalue of the determinant of the coefficients of  $C_n$  of Eqs. (117) yields the critical pressure for the case when also the  $v = 0$  boundary condition is fulfilled.

The above solution can be expressed in a more general form with the aid of Lagrangian multipliers.

Since now not all the coefficients  $C_n$  are arbitrary, substitution of Eqs. (43), (81) and (82) into Eq. (63) yields

$$\sum_{n=1}^N \sum_{m=1}^N C_n T(n, m) \delta C_m = 0 \quad (118)$$

To fulfil the boundary condition  $v = 0$ , the coefficients  $C_n$  and their variations must be related according to Eqs. (114), or

$$\begin{aligned}\sum_{m=1}^N B_{im} \delta C_m &= 0 \\ \sum_{m=1}^N B_{jm} \delta C_m &= 0\end{aligned}\tag{119}$$

Multiplication of Eqs. (119) by Lagrangian multipliers and addition to Eq. (118) yields

$$\sum_{n=1}^N \sum_{m=1}^N C_n T(n, m) \delta C_m + \lambda_i \sum_{m=1}^N B_{im} \delta C_m + \lambda_j \sum_{m=1}^N B_{jm} \delta C_m = 0\tag{120}$$

and then

$$\sum_{m=1}^N \delta C_m \left[ \sum_{n=1}^N T(n, m) C_n + \lambda_i B_{im} + \lambda_j B_{jm} \right] = 0\tag{121}$$

By proper choice of the Lagrangian multipliers  $\lambda_i$  and  $\lambda_j$  the coefficients of  $\delta C_{N-1}$  and  $\delta C_N$  will vanish. Since the remaining variations  $\delta C_m$  are independent, their coefficients must also vanish. Thus, one may consider all the variations  $\delta C_m$  in Eq. (121) arbitrary. Hence, Eqs. (121) and (114) yield

$$\begin{aligned}\sum_{n=1}^N C_n T(n, m) + \lambda_i B_{im} + \lambda_j B_{jm} &= 0 \\ \sum_{n=1}^N B_{in} C_n &= 0 \\ \sum_{n=1}^N B_{jn} C_n &= 0\end{aligned}\tag{122}$$

( $m = 1$  to  $N$ )

These are  $N + 2$  linear algebraic equations with  $n + 2$  unknowns:  $C_n$  ( $n = 1$  to  $N$ )  $\lambda_i$  and  $\lambda_j$ .

Again, the lowest eigenvalue of the determinant of the coefficients of  $C_n$ ,  $\lambda_1$  and  $\lambda_2$ , of Eqs. (122) yields the critical pressure, for a shell which fulfils also the boundary condition of zero circumferential displacement.

#### 14. SOLUTION

By substitution of Eqs. (43) into Eq. (63) and taking into account that the first two "corrected" stability equations are satisfied by the displacements Eqs. (81), one obtains

$$\begin{aligned}
 0 = \delta U = & - \int_0^{2\pi} \int_1^2 \{ [Eh/a(1-\nu^2)] [\mu_1 u_{,xx}/x - \mu_1 u_{,xx}/k_1 - \chi_1 w_{,xxx}/x + \chi_1 x w_{,xxx}/k_2^2 + \chi_2 w_{,x}/x^2 \\
 & - \chi_2 w_{,x}/k_3 x + \chi_2 w_{,\phi\phi}/x^3 \sin^2 \alpha - \chi_2 w_{,\phi\phi}/k_4 x^2 \sin^2 \alpha] \delta u \\
 & + [Eh/a(1-\nu^2)] [-\chi_2 w_{,\phi\phi\phi}/x^3 \sin^3 \alpha + \chi_2 w_{,\phi\phi\phi}/k_5 x^2 \sin^3 \alpha - \chi_2 w_{,x\phi}/x^2 \sin \alpha + \chi_2 w_{,x\phi}/k_6 x \sin \alpha] \delta v \\
 & - (D/a^3) [w_{,xxxx} + 2w_{,xxx}/x - w_{,xx}/x^2 + w_{,x}/x^3 - 2w_{,x\phi\phi}/x^3 \sin^2 \alpha + 2w_{,xx\phi\phi}/x^2 \sin^2 \alpha \\
 & + 4w_{,\phi\phi}/x^4 \sin^2 \alpha + w_{,\phi\phi\phi\phi}/x^4 \sin^4 \alpha + \cot \alpha \, 12(a^2/h^2)(-\nu u_{,x}/x - u/x^2 - v_{,\phi}/x^2 \sin \alpha \\
 & + \cot \alpha \, w/x^2) + \mu_2 \cot \alpha \, 12(a^2/h^2)(-u/x^2 - v_{,\phi}/x^2 \sin \alpha + \cot \alpha \, w/x^2) \\
 & + \eta_{01} w_{,xxxx}/x + \zeta_1 (-u_{,xxx}/x) + \eta_{02} (w_{,\phi\phi\phi\phi}/x^4 \sin^4 \alpha + 2w_{,\phi\phi}/x^4 \sin^2 \alpha \\
 & + w_{,x}/x^3 - w_{,xx}/x^2) + \zeta_2 (u_{,x}/x^2 - u/x^3 - u_{,\phi\phi}/x^3 \sin^2 \alpha + v_{,\phi x}/x^2 \sin \alpha \\
 & - v_{,\phi}/x^3 \sin \alpha - v_{,\phi\phi\phi}/x^3 \sin^3 \alpha + \cot \alpha \, w/x^3 + 2 \cot \alpha \, w_{,\phi\phi}/x^3 \sin^2 \alpha) + \eta_{1t} (w_{,xxx\phi\phi}/x^3 \sin^2 \alpha \\
 & - 2w_{,x\phi\phi}/x^4 \sin^2 \alpha + 2w_{,\phi\phi}/x^5 \sin^2 \alpha) + \eta_{2t} (w_{,xx\phi\phi}/x^2 \sin^2 \alpha - w_{,x\phi\phi}/x^3 \sin^2 \alpha \\
 & + w_{,\phi\phi}/x^4 \sin^2 \alpha) + (p a^3 \tan \alpha / D) (x w_{,xx}/2 + w_{,x} \\
 & + w_{,\phi\phi}/x \sin^2 \alpha) \} \delta w \, [a^3 x \sin \alpha \, dx d\phi]
 \end{aligned}$$

$$\begin{aligned}
& + \int_0^{2\pi} \{ [Eh/(1-\nu^2)] ax [(1 + \mu_1/x) u_{,x} + \nu(v_{,\phi}/x \sin \alpha + u/x - \cot \alpha w/x) - \chi_1 w_{,xx}/x] a \delta u \\
& + [Eh/2(1+\nu)] ax [v_{,x} - v/x + u_{,\phi}/x \sin \alpha] a \delta v \\
& + Dx[(1 + \eta_{01}/x) w_{,xx} + \nu(w_{,x}/x + \nu w_{,\phi\phi}/x^2 \sin^2 \alpha) - \zeta_1 u_{,x}/x] (\delta w)_{,x} \Big|_{x=1}^{x=x_2} \sin \alpha d\phi \quad (123)
\end{aligned}$$

Equations (81) and (82) are then substituted into Eq. (123). Since all  $\delta C_n$  are arbitrary, one can take all  $\delta C_n$  to be zero except one,  $\delta C_m$ . Then, after integration and division by a constant,  $\pi D \sin \alpha$ , one obtains

$$\begin{aligned}
& \sum_{n=1}^N C_n \{ 12(a/h)^2 [R^{(1)}(n, m) + R^{(2)}(n, m) + R^{(3)}(n, m)] \\
& + [\eta_{01} R^{\eta_{01}}(n, m) + \zeta_1 R^{\zeta_1}(n, m) + \eta_{1t} R^{\eta_{1t}}(n, m) + \zeta_2 R^{\zeta_2}(n, m) \\
& + \eta_{2t} R^{\eta_{2t}}(n, m) + R^{0\eta_{02}}(n, m) + 12(a/h)^2 \cot \alpha R^{h\mu_2}(n, m) + \lambda_p R^p(n, m)] \\
& + [12(a/h)^2 R^{N_x}(n, m) + 6(1-\nu)(a/h)^2 R^{N_x\phi}(n, m) + R^{M_x}(n, m)] \} = 0 \\
& (m = 1 \text{ to } N) \quad (124)
\end{aligned}$$

where all the terms are non-dimensional,

$$\lambda_p = p a^3 \tan \alpha / D$$

$$\text{and } D = Eh^3 / 12(1 - \nu^2) \quad (125)$$

The first part of Eq. (124) represents the virtual work done by the "error-loads" of the first two stability equations. The expression included in the second square brackets represents the virtual work of the third stability equation, and the expression in third square brackets - the "boundary work". The

superscripts of the R's indicate which part of the virtual work they represent. For example,  $R^{N_x}(n,m)$  represents the virtual work done by the longitudinal force  $N_x$  at the boundaries.

If in Eq. (124) the expression included in the curled brackets is denoted  $T(n,m)$ , the equation may be written as

$$\begin{aligned}
 C_1 T(1,1) + C_2 T(2,1) + \dots + C_{N-1} T(N-1,1) + C_N T(N,1) &= 0 \\
 C_1 T(1,2) + C_2 T(2,2) + \dots + C_{N-1} T(N-1,2) + C_N T(N,2) &= 0 \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 C_1 T(1,N-1) + C_2 T(2,N-1) + \dots + C_{N-1} T(N-1,N-1) + C_N T(N,N-1) &= 0 \\
 C_1 T(1,N) + C_2 T(2,N) + \dots + C_{N-1} T(N-1,N) + C_N T(N,N) &= 0 \quad (126)
 \end{aligned}$$

The lowest eigenvalue of the determinant of the coefficients of  $C_n$  yields again the critical pressure for general instability

$$|T(n, m)| = 0 \quad (127)$$

The integral value of  $t$  (the number of circumferential waves) which makes  $p_{cr}$  a minimum must be used in calculations.

The  $R$  functions of Eqs. (124) are defined as follows:

$$\begin{aligned}
 R^{(1)}(n, m) = I_1^{2\gamma-1}(n, m) [A_{rm} F_1^{1A}(n) + B_{rm} F_1^{1B}(n)] + I_2^{2\gamma-1}(n, m) [A_{rm} F_2^{1A}(n) + B_{rm} F_2^{1B}(n)] \\
 + I_2^{2\gamma-1}(m, n) [A_{im} F_1^{1A}(n) + B_{im} F_1^{1B}(n)] + I_3^{2\gamma-1}(n, m) [A_{im} F_2^{1A}(n) + B_{im} F_2^{1B}(n)]
 \end{aligned}$$

$$\begin{aligned}
R^{(2)}(n, m) &= I_1^{2\gamma-2}(n, m) [A_{rm} F_1^{2A}(n) + B_{rm} F_1^{2B}(n)] + I_2^{2\gamma-2}(n, m) [A_{rm} F_2^{2A}(n) + B_{rm} F_2^{2B}(n)] \\
&\quad + I_2^{2\gamma-2}(m, n) [A_{im} F_1^{2A}(n) + B_{im} F_1^{2B}(n)] + I_3^{2\gamma-2}(n, m) [A_{im} F_2^{2A}(n) + B_{im} F_2^{2B}(n)] \\
R^{(3)}(n, m) &= I_1^{2\gamma-3}(n, m) A_{rm} F_1^{3A}(n) + I_2^{2\gamma-3}(n, m) A_{rm} F_2^{3A}(n) \\
&\quad + I_2^{2\gamma-3}(m, n) A_{im} F_1^{3A}(n) + I_3^{2\gamma-3}(n, m) A_{im} F_2^{3A}(n)
\end{aligned} \tag{128}$$

$$\begin{aligned}
R^{\eta 01}(n, m) &= I_1^{2\gamma-4}(n, m) F_1^{\eta 01}(n) + I_2^{2\gamma-4}(n, m) F_2^{\eta 01}(n) \\
R^{\zeta 1}(n, m) &= I_1^{2\gamma-3}(n, m) F_1^{\zeta 1}(n) + I_2^{2\gamma-3}(n, m) F_2^{\zeta 1}(n) \\
R^{\eta 11}(n, m) &= I_1^{2\gamma-4}(n, m) F_1^{\eta 11}(n) + I_2^{2\gamma-4}(n, m) F_2^{\eta 11}(n) \\
R^{\zeta 2}(n, m) &= I_1^{2\gamma-2}(n, m) F_1^{\zeta 2}(n) + I_2^{2\gamma-2}(n, m) F_2^{\zeta 2}(n) \\
R^{\eta 12}(n, m) &= I_1^{2\gamma-3}(n, m) F_1^{\eta 12}(n) + I_2^{2\gamma-3}(n, m) F_2^{\eta 12}(n) \\
R^{0\eta 02}(n, m) &= I_1^{2\gamma-3}(n, m) F_1^{0\eta 02}(n) + I_2^{2\gamma-3}(n, m) F_2^{0\eta 02}(n) \\
R^{h\mu 2}(n, m) &= I_1^{2\gamma-1}(n, m) F_1^{h\mu 2}(n) + I_2^{2\gamma-1}(n, m) F_2^{h\mu 2}(n) \\
R^p(n, m) &= I_1^{2\gamma}(n, m) F_1^p(n) + I_2^{2\gamma}(n, m) F_2^p(n) \\
R^{N_x}(n, m) &= A_{im} [(-1)^{m+n} x_2^{2\gamma} F_2^{N_x}(n) - F_1^{N_x}(n)] \\
R^{N_x \phi}(n, m) &= B_{im} [(-1)^{m+n} x_2^{2\gamma} - 1] F^{N_x \phi}(n) \\
R^{M_x}(n, m) &= m\beta [(-1)^{m+n} x_2^{2\gamma-2} F_2^{M_x}(n) - F_1^{M_x}(n)]
\end{aligned} \tag{129}$$

(130)



$$\begin{aligned}
F_1^{1A}(n) &= \mu_{11} A_{cn} - \chi_{12} a_{3n} + \chi_{23} \gamma - \chi_{24} t^2 / \sin^2 \alpha \\
F_1^{1B}(n) &= \chi_{25} t^3 / \sin^3 \alpha - \chi_{26} \gamma t / \sin \alpha \\
F_2^{1A}(n) &= \mu_{11} A_{dn} - \chi_{12} b_{3n} + \chi_{23} n \beta \\
F_2^{1B}(n) &= -\chi_{26} n \beta t / \sin \alpha
\end{aligned} \tag{131}$$

$$\begin{aligned}
F_1^{2A}(n) &= -\mu_1 A_{cn} - \chi_2 \gamma + \chi_2 t^2 / \sin^2 \alpha \\
F_1^{2B}(n) &= -\chi_2 t^3 / \sin^3 \alpha + \chi_2 \gamma t / \sin \alpha \\
F_2^{2A}(n) &= -\mu_1 A_{dn} - \chi_2 n \beta \\
F_2^{2B}(n) &= +\chi_2 n \beta t / \sin \alpha
\end{aligned} \tag{132}$$

$$\begin{aligned}
F_1^{3A}(n) &= \chi_1 a_{3n} \\
F_2^{3A}(n) &= \chi_1 b_{3n}
\end{aligned} \tag{133}$$

$$\begin{aligned}
F_1^{\eta 01}(n) &= \gamma(\gamma-1)(\gamma-2)(\gamma-3) + n^2 \beta^2 [n^2 \beta^2 - 11 + 6\gamma(3-\gamma)] \\
F_2^{\eta 01}(n) &= 2n\beta [n^2 \beta^2 (3-2\gamma) + 2\gamma^3 - 9\gamma^2 + 11\gamma - 3]
\end{aligned} \tag{134}$$

$$\begin{aligned}
F_1^{\zeta 1}(n) &= -A_{rn} a_{3n} + A_{in} b_{3n} \\
F_2^{\zeta 1}(n) &= -A_{rn} b_{3n} - A_{in} a_{3n}
\end{aligned} \tag{135}$$

$$F_1^{\eta^{t1}}(n) = (t^2/\sin^2 a) (-\gamma^2 + 3\gamma - 2 + n^2 \beta^2)$$

$$F_2^{\eta^{t1}}(n) = (t^2/\sin^2 a) (3 - 2\gamma) n \beta \quad (136)$$

$$F_1^{\zeta^2}(n) = A_{rn} (\gamma + t^2/\sin^2 a - 1) + A_{in} (-n\beta) + B_{rn} [(t/\sin a) (1-\gamma) - t^3/\sin^3 a]$$

$$+ B_{in} n\beta (t/\sin a) + \cot a (1 - 2t^2/\sin^2 a)$$

$$F_2^{\zeta^2}(n) = A_{rn} n\beta + A_{in} (t^2/\sin^2 a + \gamma - 1) + B_{rn} n\beta (-t/\sin a)$$

$$+ B_{in} [-t^3/\sin^3 a + (t/\sin a) (1-\gamma)] \quad (137)$$

$$F_1^{\eta^{t2}}(n) = (t^2/\sin^2 a) (-\gamma^2 + 2\gamma - 1 + n^2 \beta^2)$$

$$F_2^{\eta^{t2}}(n) = (t^2/\sin^2 a) (2 - 2\gamma) n \beta \quad (138)$$

$$F_1^{0\eta^{02}}(n) = (t^4/\sin^4 a) (1 + \eta_{02}) + (2t^2/\sin^2 a) (n^2 \beta^2 - \eta_{02} - 2 + 2\gamma - \gamma^3) + n^4 \beta^4$$

$$+ n^2 \beta^2 (\eta_{02} - 4 + 12\gamma - 6\gamma^2) + \eta_{02} (2\gamma - \gamma^2) + 4\gamma^2 - 4\gamma^3 + \gamma^4$$

$$F_2^{0\eta^{02}}(n) = 4n\beta [(t^2/\sin^2 a) (1-\gamma) + n^2 \beta^2 (1-\gamma) + (\eta_{02}/2) (1-\gamma) + 2\gamma - 3\gamma^2 + \gamma^3] \quad (139)$$

$$F_1^{h\mu^2}(n) = A_{rn} [-\nu\gamma - (1 + \mu_2)] + A_{in} \nu n\beta + B_{rn} (t/\sin a) (1 + \mu_2) + \cot a (1 + \mu_2)$$

$$F_2^{h\mu^2}(n) = A_{rn} (-\nu n\beta) + A_{in} [-\nu\gamma - (1 + \mu_2)] + B_{in} (t/\sin a) (1 + \mu_2) \quad (140)$$

$$F_1^p(n) = -t^2/\sin^2 a - (1/2) (n^2 \beta^2 - \gamma - \gamma^2)$$

$$F_2^p(n) = (1/2) n\beta (1 + 2\gamma) \quad (141)$$

$$\begin{aligned}
F_1^{N_x}(n) &= (A_{in} \gamma + A_{rn} n \beta) (1 + \mu_1) - \nu [B_{in} (t/\sin \alpha) - A_{in}] + \chi_1 n \beta (1 - 2\gamma) \\
F_2^{N_x}(n) &= (A_{in} \gamma + A_{rn} n \beta) (1 + \mu_1/x_2) - \nu [B_{in} (t/\sin \alpha) - A_{in}] + \chi_1 n \beta (1 - 2\gamma)/x_2^2 \\
F^{N_x\phi}(n) &= -B_{in} (1 - \gamma) + B_{rn} n \beta + A_{in} (t/\sin \alpha) \\
F_1^{M_x}(n) &= n \beta [(2\gamma - 1) (1 + \eta_{01}) + \nu] - \zeta_1 (A_{in} \gamma + A_{rn} n \beta) \\
F_2^{M_x}(n) &= n \beta [(2\gamma - 1) (1 + \eta_{01}/x_2) + \nu] - \zeta_1 (A_{in} \gamma + A_{rn} n \beta)
\end{aligned} \tag{142}$$

## 15. APPROXIMATE SOLUTION BY NEGLECT OF ECCENTRICITY OF THE STIFFENERS

For conical shells, stiffened by equal and equally spaced frames, a simple approximate method for calculation of the critical pressure can be derived. Calculations of the critical pressure for ring-stiffened conical and cylindrical shells (see Ref. 13), have shown, that when the eccentricity of the stiffeners is neglected,  $p_{cr}$  is between that for internal rings and that for external ones. Hence, for the purpose of an approximation, the following assumptions may be made for a ring stiffened shell: the extentional stiffness of the shell is increased by that of the rings, and the total bending stiffness in the circumferential direction is taken to be the stiffness of the combined cross-section. The torsional stiffness of the ring is assumed to be small and hence negligible. With these assumptions, the internal forces and moments given in Eqs. (36) and (37) become:

$$\begin{aligned}
N_x &= [Eh/(1 - \nu^2)] [\epsilon_x + \nu \epsilon_\phi] \\
N_\phi &= [Eh/(1 - \nu^2)] [(1 + \mu_2) \epsilon_\phi + \nu \epsilon_x] \\
N_{x\phi} = N_{\phi x} &= [Eh/2 (1 + \nu)] \gamma_{x\phi}
\end{aligned} \tag{143}$$

$$\begin{aligned}
M_x &= -(D/a) (\kappa_x + \nu \kappa_\phi) \\
M_\phi &= -(D/a) [(1 + \delta_2 \eta_{02}) \kappa_\phi + \nu \kappa_x] \\
M_{x\phi} &= -M_{\phi x} = (D/a) (1 - \nu) \kappa_x \phi
\end{aligned} \tag{144}$$

If one denotes

$$\delta_2 \eta_{02} = \eta_2 \tag{145}$$

and equates the circumferential bending stiffness given in Eqs. (144) with that of the combined cross-section, one obtains

$$D(1 + \eta_2) = (1/a_0) [E_2 I_{22} + E_2 A_2 (e_2 - \bar{z}_2)^2 + a_0 D + E a_0 h \bar{z}_2^2 / (1 - \nu^2)] \tag{146}$$

and hence

$$\eta_2 = 12(1 - \nu^2) (E_2/E) \{ I_{22}/a_0 h^3 + (A_2/ah) [(e_2 - \bar{z}_2)/h]^2 \} + 12 (\bar{z}_2/h)^2 \tag{147}$$

With the simplified force and moment expressions of Eqs. (143) and (144) no "correcting coefficients" are needed. In all the formulae, the terms introduced by the stiffeners, except  $\mu_2$  and  $\eta_{02}$ , vanish; and of the two exceptions  $\eta_{02}$  is replaced by  $\eta_2$  from Eq. (147).

A further simplification is possible if one neglects  $\mu_2$ . Bodner (Ref. 9) showed on mathematical grounds that neglecting of  $\mu_2$  introduces an error of less than 1 percent in  $p_{cr}$ . Calculations for typical cylindrical and conical shells carried out in connection with the present work verified that the error is much smaller than 1 percent in both types of shells. Furthermore, the calculations showed, that the approximation involved by neglecting  $\mu_2$  is of much smaller magnitude than the neglect of the effect of eccentricity of stiffeners inherent in the approach of this section.

Hence, in the approximate solution proposed here for ring-stiffened conical shells under hydrostatic pressure, the following substitutions must be made in all formulae:

$$\begin{array}{llll}
\mu_1 \rightarrow 0 & \mu_2 \rightarrow 0 & & \\
\chi_1 \rightarrow 0 & \chi_2 \rightarrow 0 & & \\
\zeta_1 \rightarrow 0 & & \eta_{02} \rightarrow \eta_2 & \\
\eta_{t1} \rightarrow 0 & \zeta_2 \rightarrow 0 & & \\
\eta_{01} \rightarrow 0 & \eta_{t2} \rightarrow 0 & & 
\end{array} \tag{148}$$

With Eqs. (148) the calculation of  $p_{cr}$  becomes much easier, since all the terms involving "correcting coefficients" are multiplied by  $\mu_1$ ,  $\chi_1$  or  $\chi_2$  which are assumed to be zero.

The final results of the above approximate solution were also obtained (at the same time), independently, in Ref. 10, by another approach. The method applied there, was substitution of the stiffened conical shell by an equivalent orthotropic one. One very small difference should, however, be noted: in Ref. 10 the increase of the shell cross-section due to the rings (when it is taken into account) is not multiplied by  $(1 - \nu^2)$  whereas here  $\mu_2$ , which stems directly from Eqs. (38), contains this multiplication already.

Although the same final results are obtained by the equivalent orthotropic shell approach and by that of this section, the two approaches differ considerably. The orthotropic approach requires orthogonal shell properties, which imply equal and equally spaced rings and stringers (if any) which vary according to the cone radius; whereas in the approximate method of this section, as well as in the preceding more accurate method, no similar restrictions are implied.

Here again, the additional stiffnesses due to the stiffeners can be some functions of  $x$  (the stiffeners have to be symmetrical with respect to the cone axis).

If  $\mu_1$  and  $\mu_2$  and the eccentricity of the stiffeners are neglected, the internal forces and moments become

$$N_x = [Eh/(1 - \nu^2)] (\epsilon_x + \nu \epsilon_\phi)$$

$$N_\phi = [Eh/(1 - \nu^2)] (\epsilon_\phi + \nu \epsilon_x)$$

$$N_x \phi = N_{\phi x} = [Eh/2(1+\nu)] \gamma_x \phi \quad (149)$$

$$M_x = (-D/a) \{ \kappa_x [1 + \bar{\eta}_1(x)] + \nu \kappa_\phi \}$$

$$M_\phi = (-D/a) \{ \kappa_\phi [1 + \bar{\eta}_2(x)] + \nu \kappa_x \}$$

$$M_x \phi = -M_{\phi x} = (D/a) (1 - \nu) \kappa_x \phi \quad (150)$$

It should be noted that  $\bar{\eta}_1$  is a function of  $x$  even when the cross section of the stringers does not vary. For constant area stringers, one obtains

$$D[1 + \bar{\eta}_1(x)] = (1/b_0 x) [E_1 I_{11} + E_1 A_1 (e_1 - \bar{z}_1)^2 + E b_0 x h \bar{z}_1^2 / (1 - \nu^2) + b_0 x D] \quad (151)$$

and then

$$\bar{\eta}_1(x) = (1/b_0 D x) [E_1 I_{11} + E_1 A_1 (e_1 - \bar{z}_1)^2 + E b_0 x h \bar{z}_1^2 / (1 - \nu^2)] \quad (152)$$

From Eqs. (36) one obtains directly

$$\bar{\eta}_1(x) = \delta_1 \eta_{01}/x = \eta_1/x \quad (153)$$

Two similar, but not equal, functions for  $\bar{\eta}_1$  are obtained. The expression in the square brackets of Eq. (152) varies only slightly with  $x$ . For example, for  $A_1/b_0 h = 0.1471$ ,  $e_1/h = 1.653$ ,  $I_{11}/b_0 h^3 = 0.7819$  and  $E_1 = E$  this value is 7% larger than that at  $x = 1$  when  $x = 6$ , or 11% larger when  $x \rightarrow \infty$ .  $\bar{\eta}_1(x)$  defined in Eq. (153) is the more accurate function, since it stems directly from Eqs. (36). Eqs. (152) and Eq. (153) can be exactly equal only at a particular value of  $x$ , say at the midheight of the conical shell, where

$$\bar{x} = (1 + x_2)/2 \quad (154)$$

then  $\eta_1$  becomes

$$\eta_1 = 12(1 - \nu^2)(E_1/E) \{ I_{11}/b_0 h^3 + (A_1/b_0 h) [(e_1 - \bar{z}_1)/h]^2 \} + 12(\bar{z}_1/h)^2 \bar{x} \quad (155)$$

For a conical shell stiffened by stringers only, all the terms due to the stiffeners vanish, except  $\eta_{01}$  which must be replaced by  $\eta_1$  from Eq. (155).

Since the effect of stringers on  $p_{cr}$  is rather small, and the effect of their eccentricity is of the same magnitude, the calculation of  $p_{cr}$  without consideration of the eccentricity has little value. Hence Eq. (155) will be of limited use for stringer-stiffened conical shells under hydrostatic pressure. It may however be useful in other loading cases.

## 16. EFFECTIVE LENGTH OF SHEET

In the previous sections it has been assumed that the stiffeners are closely spaced and therefore the entire shell is active. If the distance between the stiffeners is larger than a certain magnitude, only part of the sheet between the stiffeners is active. The resulting decrease in the total stiffness of the shell usually expressed as "effective length of the sheet" can also be expressed as a decrease in the modulus of elasticity of the stiffeners.

If the circumferential stiffness of the combined cross-section for a wholly active sheet but "effective moduli" of stiffeners is compared with that for an "effective length" -  $a_e$  - one obtains.

$$E_2 [I_{22} + A_2 (e_2 - \bar{z}_2)^2] + E [a_0 h^3 / 12(1 - \nu^2) + a_0 h \bar{z}_2^2 / (1 - \nu^2)] = E'_2 [I_{22} + A_2 (e_2 - \bar{z}_2')^2] + E [a_e h^3 / 12(1 - \nu^2) + a_e h \bar{z}_2'^2 / (1 - \nu^2)] \quad (156)$$

and then

$$E_2/E = \frac{12(1 - \nu^2)(E'_2/E) \{ I_{22}/a_0 h^3 + (A_2/a_0 h) [(e_2 - \bar{z}_2')/h]^2 \} + a_e/a_0 - 1 + 12(\bar{z}_2'/h)^2 (a_e/a_0) - 12(\bar{z}_2/h)^2}{12(1 - \nu^2) \{ I_{22}/a_0 h^3 + (A_2/a_0 h) [(e_2 - \bar{z}_2)/h]^2 \}} \quad (157)$$

In the longitudinal direction it is assumed that the "effective length" varies linearly in the same manner as the actual distance between the stringers. The respective cross-sections are equated at the midheight of the conical shell.

Then

$$E_1/E = \frac{12(1-\nu^2)(E'_1/E)\{I_{11}/b_0h^3 + (A_1/b_0h)[(e_1 - \bar{z}'_1)/h]^2\} + \bar{x}[b_e/b_0 - 1 + 12(\bar{z}'_1/h)^2(b_e/b_0) - 12(\bar{z}_1/h)^2]}{12(1-\nu^2)\{I_{11}/b_0h^3 + (A_1/b_0h)[(e_1 - \bar{z})/h]^2\}} \quad (158)$$

In Eqs. (156) to (158),  $a_e$  and  $b_e$  are "effective lengths",  $E_1$  and  $E_2$  are "effective" moduli of elasticity of the stiffeners,  $E'_1$  and  $E'_2$  are actual moduli of elasticity of stiffeners,  $\bar{z}_1$  and  $\bar{z}_2$  are the distances of the overall centroid of the stiffener-shell combination from the middle surface when the sheet length is  $a_0$  or  $b_0\bar{x}$  respectively, and  $\bar{z}'_1$  and  $\bar{z}'_2$  are the distances of the overall centroid of the stiffener-shell combination from the middle surface, when the sheet length is  $a_e$  or  $b_e\bar{x}$  respectively.

## 17. APPROXIMATE FORMULAE FOR RING STIFFENED SHELLS.

In Ref. 10 it is shown that the ratio between the critical pressure  $p_{cr}$  of an orthotropic conical shell and that of its equivalent cylindrical shell,  $\bar{p}_{cr}$  may be approximated by the same ratio for isotropic shells.

$$p_{cr}/\bar{p}_{cr} = g(\psi)$$

$$\psi = 1 - (R_1/R_2) = 1 - (1/x_2) \quad (159)$$

The function  $g(\psi)$  is given in Ref. 1 and Ref. 11. It is reproduced here in a tabular form (Table 3), the values having been read off the curve in Ref. 11. It was pointed out in Ref. 10 that the accuracy of  $g(\psi)$  diminishes slightly when  $\alpha$  is greater than 45%.



T A B L E 3

$\psi$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
g	1.00	1.005	1.01	1.02	1.04	1.075	1.13	1.19	1.22	1.21	1.175

The equivalent cylindrical shell is taken as one having a length equal to the slant length of the cone,  $\ell$ , a radius equal to its average radius of curvature  $\rho_{av}$ , and the same thickness and ring-stiffeners. Based on the results of Niordson (Ref. 12) and Bodner (Ref. 9), the critical pressure for an equivalent cylindrical shell may be written

$$\bar{p}_{cr}/E = [1/(t_0^2 + 0.5 c_0^2)] \{ (h/\rho_{av}) [c_0^2/(t_0^2 + c_0^2)]^2 + [h^3/12(1-\nu^2) \rho_{av}^3] [(t_0^2 + c_0^2)^2 + \eta_2 t_0^4] \} \quad (160)$$

where

$$t_0 = t / \cos \alpha$$

$$\rho_{av} = (a \tan \alpha / 2) (1 + x_2) = (R_1 + R_2) / 2 \cos \alpha$$

$$c_0 = \pi \rho_{av} / \ell = (\pi \tan \alpha / 2) [(x_2 + 1) / (x_2 - 1)]$$

$$\ell = a(x_2 - 1) = (R_2 - R_1) / \sin \alpha \quad (161)$$

In general  $c_0$  is much smaller than  $t_0$  and then Eq. (160) may be written in a more convenient form by changing the first denominator from  $(t_0^2 + 0.5 c_0^2)$  to  $(t_0^2 + c_0^2)$

$$\bar{p}_{cr}/E \sim [1/(t_0^2 + c_0^2)] \{ (h/\rho_{av}) [c_0^2/(t_0^2 + c_0^2)]^2 + [h^3/12(1-\nu^2) \rho_{av}^3] [(t_0^2 + c_0^2)^2 + \eta_2 t_0^4] \} \quad (162)$$

This change reduces the critical pressure slightly. Its physical interpretation is the conservative assumption that twice the actual hydrostatic pressure acts upon the rigid bulkheads closing the equivalent

cylindrical shell. Since, however, in case of hydrostatic pressure, the effect of the pressure acting in the longitudinal direction is small compared to that acting radially, Eq. (162) may be expected to yield a good approximation.

One can calculate the minimum value of  $\bar{p}_{cr}$  from Eq. (162), by assuming  $p_{cr}$  to be a continuous function of  $(t_0^2 + c_0^2)$ . Only very slight conservative errors are involved in this assumption. Hence, with the notation

$$t_0^2 + c_0^2 = S \quad (163)$$

One obtains for a ring-stiffened conical shell

$$p_{cr}/E = [h^3/\rho_{av}^3 12(1-\nu^2)][(1+\eta_2)S - 2\eta_2 c_0^2 + \eta_2 c_0^4/S + (\rho_{av}/h)^2 12(1-\nu^2)(c_0^4/S^3)] g(\psi) \quad (164)$$

$p_{cr}$  is a minimum when

$$S = t_0^2 + c_0^2 = c_0 \{ \eta_2 c_0^2 + [\eta_2 c_0^4 + 144(1-\nu^2)(1+\eta_2)(\rho_{av}/h)^2]^{0.5} \}^{0.5} [2(1+\eta_2)]^{-0.5} \quad (165)$$

Since  $c_0$  is usually much smaller than  $t_0$ , one could alternatively approximate Eq. (160) by neglecting  $c_0^2$  in comparison with  $t_0^2$

$$p_{cr}/E \sim (1/t_0^2) [h c_0^4/\rho_{av} t_0^4 + h^3 t_0^4 (1+\eta_2)/\rho_{av}^3 12(1-\nu^2)] g(\psi) \quad (166)$$

If  $p_{cr}$  is again assumed to be a continuous function of  $t_0$ , it is a minimum when

$$t_0^2 = c_0 \sqrt{6} (\rho_{av}/h)^{0.5} (1-\nu^2)^{0.25} (1+\eta_2)^{-0.25} = (1-\nu^2)^{0.25} \pi \sqrt{6} (\rho_{av}/\ell) (\rho_{av}/h)^{0.5} (1+\eta_2)^{-0.25} \quad (167)$$

and this minimum value is

$$p_{cr}/E = [\sqrt{6} \pi / 9 (1-\nu^2)^{0.75}] (\rho_{av}/\ell) (h/\rho_{av})^{2.5} (1+\eta_2)^{0.75} g(\psi) \quad (168)$$

It should be noted that whereas Eq. (164) yields a low value for  $p_{cr}$ , a high value is obtained from Eq. (168), since there the effect of the frames is slightly exaggerated.

Further simplification of Eq. (164) is possible by neglect of the small term,  $\eta_2 c_0^4/S$ . This raises  $p_{cr}$  slightly. The increase is however partly compensated by a replacement in Eq. (164) of  $S$ , from Eq. (165) by its approximation  $t_0^2$  from Eq. (167). The simplified formula obtained in this manner and that of Eq. (168) are averaged, and finally a simple formula for the critical external hydrostatic pressure of a ring-stiffened conical shell (equal rings, equally spaced) failing by general instability is obtained:

$$p_{cr}/E = [\sqrt{6}\pi/9(1-\nu^2)^{0.75}] (\rho_{av}/\ell) (h/\rho_{av})^{2.5} \{ (1+\eta_2)^{0.75} - [\sqrt{6}\pi/8(1-\nu^2)^{0.25}] (\rho_{av}/\ell) (h/\rho_{av})^{0.5} \eta_2 \} g(\psi) \quad (169)$$

If  $\eta_2 = 0$  is substituted in Eq. (169), Seide's approximate formula for unstiffened conical shells (Ref. 11, Eq. 23) is obtained.

With  $\nu = 0.3$ ,  $\sqrt{6}\pi/8(1-0.3^2)^{0.25} = 0.99 \sim 1$ , and Eq. (169) becomes

$$p_{cr}/E = 0.92 (\rho_{av}/\ell) (h/\rho_{av})^{2.5} [ (1+\eta_2)^{0.75} - (\rho_{av}/\ell) (h/\rho_{av})^{0.5} \eta_2 ] g(\psi) \quad (170)$$

If  $p_0$  is the critical pressure for an unstiffened conical shell, which was calculated with  $\eta_2 = 0$ , Eqs. (169) and (170) may be rewritten in a simpler form, where  $\sqrt{6}\pi/8(1-\nu^2)^{0.25}$  is assumed to be approximately unity for all likely values of  $\nu$ ,

$$p_{cr} = p_0 [ (1+\eta_2)^{0.75} - (\rho_{av}/\ell) (h/\rho_{av})^{0.5} \eta_2 ] \quad (171)$$

In this section, as elsewhere in this report, the work of the shear forces  $Q$  is not taken into account. The theory is a Donnell type theory, and is valid only when  $t$  (the number of circumferential waves) is greater than two (Ref. 9). Hence, the approximate formulae of this section are applicable when the  $t$ , obtained from Eqs. (165) or (167), is more than two. In the above approximate derivation it was also assumed that the semi-empirical function for  $t$  of an unstiffened conical shell, when  $\psi > 0.04$  (Fig. 8 of Ref. 11) holds also for a ring-stiffened shell.

It may be pointed out that all the above approximate formulae apply also to the limiting case of a ring-stiffened cylindrical shell. For the cylindrical shell,  $\rho_{av}$  of the above formulae is the radius of the shell,  $\ell$  its length, and  $\cos \alpha$  and  $g(\psi)$  are unity.

## 18. NUMERICAL RESULTS AND DISCUSSION

The critical pressures for general instability are computed for typical cases (Table 4). The torsional resistance of the stiffeners is neglected and the whole distance between the stiffeners is taken as the "effective length" of the sheet. The moduli of elasticity of sheet and stiffeners are equal.

Two cases are computed by all the methods proposed in this report. The results obtained are compared with those for unstiffened conical shells given in Ref. 1. Two other cases are computed only by the methods of Section 15 ( $\eta_{02} \rightarrow \eta_2$ ) and Section 17 (Eq. 169), and compared with the results obtained by Seide's approximate formula (Ref. 11) for unstiffened conical shells.

The rate of convergence of the solution is similar to that for unstiffened conical shells. For short shells a two term, or even a one term solution is sufficient. For long shells, additional terms must be considered.

The "boundary work" reduces the critical pressure slightly. Its influence upon the critical pressure is less than 1% for short shells (shells Nos. 2 to 7) but it is 3% to 4% for long ones (shells Nos. 9 to 12).

Table 4 demonstrates clearly that, as for stiffened cylindrical shells, frames are very effective in stiffening conical shells against hydrostatic pressure. For short shells (shells Nos. 2 to 5) addition of only 15% of material increases the critical pressure more than 3 times that for the similar unstiffened shell. For long shells (shells Nos. 9 to 12), this increase is larger, and for the same addition of 15% of material it is more than 3.5 times. Addition of the same material uniformly to the thickness of the shell would increase the critical pressure only 1.4 times. Addition of 22.5% of material (shells Nos. 14 and 15) increases the critical pressure more than 6 times, whereas addition of the same material to the thickness would increase the critical pressure only 1.66 times. The increase with length

in the ratio of the critical pressure for ring-stiffened conical shells to that for corresponding unstiffened ones, indicated also by the approximate formula, is due to a larger part of the hydrostatic pressure being transmitted in the circumferential direction for longer shells.

Stringers are much less effective as stiffeners against hydrostatic pressure.

The number of circumferential waves,  $t$ , for which a minimum of the critical pressure is obtained, decreases with increase in the stiffening of the shell. The order of magnitude of this decrease given by Eq. (167) and verified by the results of Table 4, is  $(1 + \eta_2)^{-0.125}$ . It should be noted, that when  $\psi > 0.64$ , the number of waves obtained from Eq. (167) is multiplied by a coefficient taken from Figure 8 of Ref. 11.

The effect of the eccentricity of the frames on the critical pressure may be summarized as follows. Internal frames yield higher general instability pressures than external frames. In a typical case, internal frames (shell No. 2) yield a critical pressure 7% greater than that obtained by external frames (shell No. 3). For long shells, the effect of the eccentricity of the frames is much more pronounced. For a typical long shell with internal frames (shell No. 9) the critical pressure is 12% higher than for the same shell with external frames (shell No. 10). This effect should be taken into account especially in the analysis of experimental results. Internal frames yield a higher critical pressure due to their smaller radius which makes them stiffer. The critical pressure obtained by the method of Section 15 ( $\eta_{02} + \eta_2$ ) is found to be somewhere between the critical pressures for internal and external frames.

For longitudinal stiffeners (stringers), the effect of eccentricity is opposite to that in frames. External stringers yield higher critical pressures than internal ones. In Table 4, external stringers of the same magnitude as that of the frames (shell No. 7) yield a critical pressure only 12% higher than that for the corresponding unstiffened shell, whereas for internal stringers (shell No. 8) the increase is only 2%. Note that the increase in critical pressure due to stringers is considerably less than that obtained by uniform thickening of the shell with the same amount of material. Hence, stringers, and especially internal stringers are very inefficient stiffeners for conical and cylindrical shells under hydrostatic pressure (see also Ref. 13). However, if stringers are taken into account one should not ignore their eccentricity. The above shown inefficiency of stringers applies only to the case of external pressure loading. For other loads, stringers are much more effective and should be subject to further investigation.

T A B L E 4

No.	S H E L L   G E O M E T R Y				(p <sub>c</sub> /E) 10 <sup>6</sup>							Eq. (169)		
					with "boundary work"			without "boundary work"						
					1 term	2 terms	3 terms	1 term	2 terms	3 terms				
1	α = 30°	Unstiffened	Singer (Ref. 1)	Internal rings	Sec. 14	e <sub>2</sub> /h = +1.653	1.315	1.290		1.313	1.286			
2	a = 57.59"						10	10		10	10			
3	x <sub>2</sub> = 1.5						0.410	0.405	0.405					
4	h = 0.1"						t = 12	12	12					
5	ν = 0.3													
6	ψ = 0.333	Rings	Sec. 14	e <sub>2</sub> /h = -1.653	e <sub>2</sub> /h = ±1.653	1.236	1.217		1.233	1.209				
7						10	10		10	10				
8	α = 30°					1.281	1.263		1.279	1.254				
9	a = 57.59"					10	10		10	10				
10	x <sub>2</sub> = 5													
11	h = 0.1"	Stringers	Sec. 14	e <sub>1</sub> /h = -1.653	e <sub>1</sub> /h = ±1.653	0.4195			0.4188					
12	ν = 0.3					12			12					
13	ψ = 0.800					0.4581			0.4581					
14						t = 8	0.0169	0.0159						
15						6	10	11						
16	α = 40°	Unstiffened	Singer (Ref. 1)	Internal rings	Sec. 14	e <sub>2</sub> /h = +1.653	0.0816	0.0615	0.0589	0.0797	0.0593	0.0580		
17	a = 77.8mm						6	7	9	6	7	9		
18	x <sub>2</sub> = 6.35						0.0717	0.0561	0.0538	0.0695	0.0533	0.0519		
19	h = 0.8mm						6	7	9	6	7	9		
20	ν = 0.33						0.0762	0.0592	0.0557	0.0742	0.0579	0.0545		
21	ψ = 0.84	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						0.0558			
22												8 - 9		
23	α = 20°												0.420	
24	a = 132mm												8 - 9	
25	x <sub>2</sub> = 3.1													
26	h = 0.64mm	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	4.28	2.93	2.61					
27	ν = 0.33						t = 4	5	6					
28	ψ = 0.84													
29														
30														
31	α = 20°	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						2.68			
32	a = 132mm											6 - 7		
33	x <sub>2</sub> = 3.1												1.30	
34	h = 0.64mm											6 - 7		
35	ν = 0.3													
36	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
37	a = 132mm						t = 4	5				4 - 5		
38	x <sub>2</sub> = 3.1													
39	h = 0.64mm													
40	ν = 0.3													
41	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
42	a = 132mm											4 - 5		
43	x <sub>2</sub> = 3.1													
44	h = 0.64mm													
45	ν = 0.3													
46	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
47	a = 132mm						t = 4	5				4 - 5		
48	x <sub>2</sub> = 3.1													
49	h = 0.64mm													
50	ν = 0.3													
51	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
52	a = 132mm											4 - 5		
53	x <sub>2</sub> = 3.1													
54	h = 0.64mm													
55	ν = 0.3													
56	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
57	a = 132mm						t = 4	5				4 - 5		
58	x <sub>2</sub> = 3.1													
59	h = 0.64mm													
60	ν = 0.3													
61	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
62	a = 132mm											4 - 5		
63	x <sub>2</sub> = 3.1													
64	h = 0.64mm													
65	ν = 0.3													
66	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
67	a = 132mm						t = 4	5				4 - 5		
68	x <sub>2</sub> = 3.1													
69	h = 0.64mm													
70	ν = 0.3													
71	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
72	a = 132mm											4 - 5		
73	x <sub>2</sub> = 3.1													
74	h = 0.64mm													
75	ν = 0.3													
76	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
77	a = 132mm						t = 4	5				4 - 5		
78	x <sub>2</sub> = 3.1													
79	h = 0.64mm													
80	ν = 0.3													
81	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
82	a = 132mm											4 - 5		
83	x <sub>2</sub> = 3.1													
84	h = 0.64mm													
85	ν = 0.3													
86	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
87	a = 132mm						t = 4	5				4 - 5		
88	x <sub>2</sub> = 3.1													
89	h = 0.64mm													
90	ν = 0.3													
91	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
92	a = 132mm											4 - 5		
93	x <sub>2</sub> = 3.1													
94	h = 0.64mm													
95	ν = 0.3													
96	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
97	a = 132mm						t = 4	5				4 - 5		
98	x <sub>2</sub> = 3.1													
99	h = 0.64mm													
100	ν = 0.3													
101	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
102	a = 132mm											4 - 5		
103	x <sub>2</sub> = 3.1													
104	h = 0.64mm													
105	ν = 0.3													
106	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
107	a = 132mm						t = 4	5				4 - 5		
108	x <sub>2</sub> = 3.1													
109	h = 0.64mm													
110	ν = 0.3													
111	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
112	a = 132mm											4 - 5		
113	x <sub>2</sub> = 3.1													
114	h = 0.64mm													
115	ν = 0.3													
116	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
117	a = 132mm						t = 4	5				4 - 5		
118	x <sub>2</sub> = 3.1													
119	h = 0.64mm													
120	ν = 0.3													
121	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
122	a = 132mm											4 - 5		
123	x <sub>2</sub> = 3.1													
124	h = 0.64mm													
125	ν = 0.3													
126	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
127	a = 132mm						t = 4	5				4 - 5		
128	x <sub>2</sub> = 3.1													
129	h = 0.64mm													
130	ν = 0.3													
131	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
132	a = 132mm											4 - 5		
133	x <sub>2</sub> = 3.1													
134	h = 0.64mm													
135	ν = 0.3													
136	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
137	a = 132mm						t = 4	5				4 - 5		
138	x <sub>2</sub> = 3.1													
139	h = 0.64mm													
140	ν = 0.3													
141	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
142	a = 132mm											4 - 5		
143	x <sub>2</sub> = 3.1													
144	h = 0.64mm													
145	ν = 0.3													
146	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
147	a = 132mm						t = 4	5				4 - 5		
148	x <sub>2</sub> = 3.1													
149	h = 0.64mm													
150	ν = 0.3													
151	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
152	a = 132mm											4 - 5		
153	x <sub>2</sub> = 3.1													
154	h = 0.64mm													
155	ν = 0.3													
156	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
157	a = 132mm						t = 4	5				4 - 5		
158	x <sub>2</sub> = 3.1													
159	h = 0.64mm													
160	ν = 0.3													
161	ψ = 0.678	Rings	Sec. 17	e <sub>2</sub> /h = ±1.653	e <sub>2</sub> /h = ±1.653						3.52			
162	a = 132mm											4 - 5		
163	x <sub>2</sub> = 3.1													
164	h = 0.64mm													
165	ν = 0.3													
166	ψ = 0.678	Unstiffened	Seide (Ref. 11)	Eccentricity neglected	Sec. 15	η <sub>02</sub> → η <sub>2</sub>	3.948	3.537				3.52		
167	a = 132mm						t = 4	5				4 - 5		
168</														

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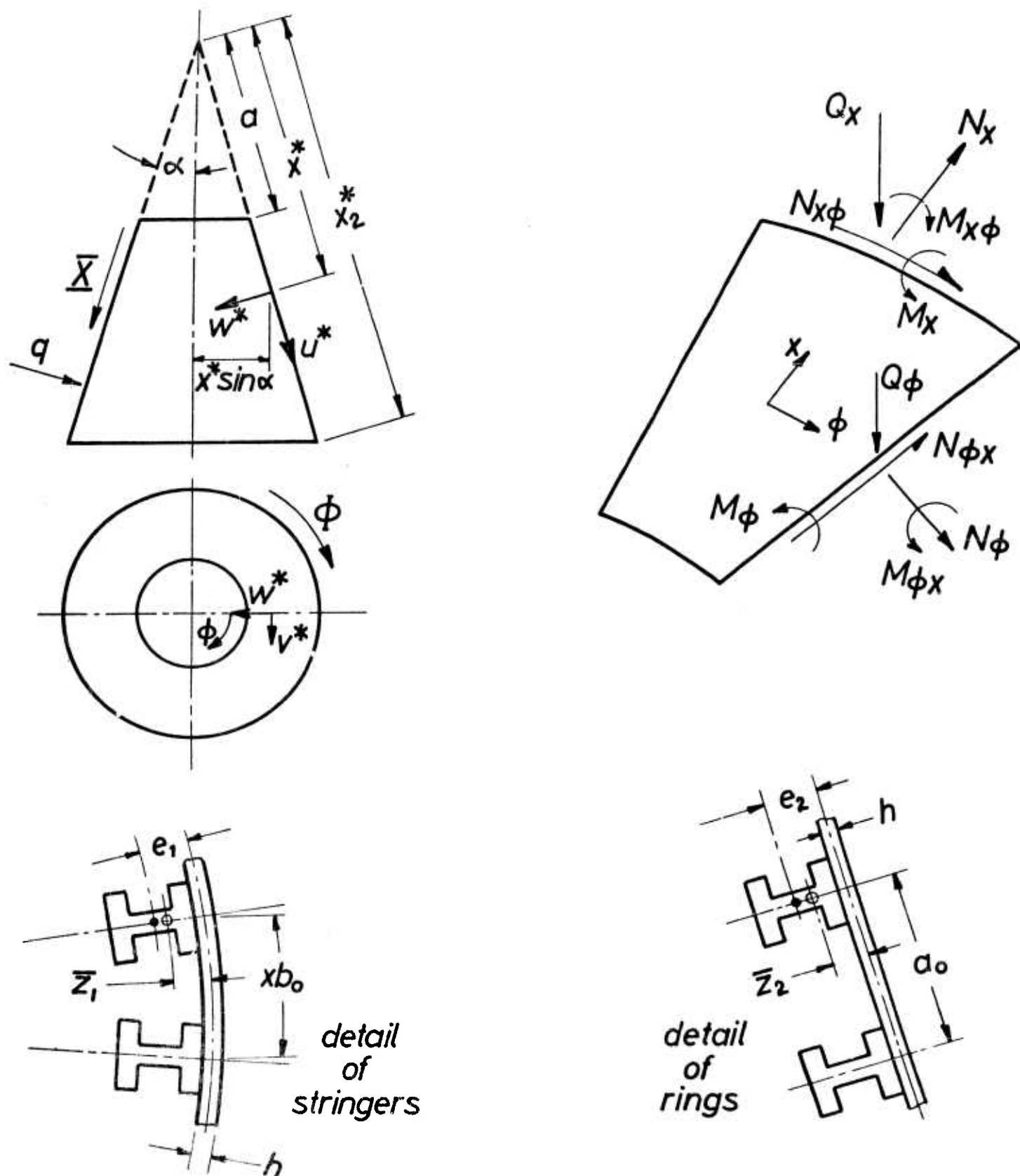


FIG. 1 NOTATION

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The stability equations are solved for general instability under hydrostatic pressure by the method of virtual displacements. The solution used earlier for unstiffened conical shells, which satisfies some of the boundary conditions of simple supports only approximately, is again applied here. The effect of this incomplete compliance with boundary conditions is shown to be negligible by consideration of

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The stability equations are solved for general instability under hydrostatic pressure by the method of virtual displacements. The solution used earlier for unstiffened conical shells, which satisfies some of the boundary conditions of simple supports only approximately, is again applied here. The effect of this incomplete compliance with boundary conditions is shown to be negligible by consideration of

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GENERAL INSTABILITY OF STIFFENED CIRCULAR CONICAL SHELLS  
UNDER HYDROSTATIC PRESSURE.

A B S T R A C T: Donnell type equilibrium and stability equations are derived for stiffened thin conical shells. The stiffeners are considered closely spaced and are therefore assumed to be "distributed" over the whole surface of the shell. In the proposed theory the stiffeners and their spacing may vary in any prescribed manner, but here only equally spaced stiffeners are dealt with. The force - and moment - strain relations of the combined stiffener-sheet cross-section are determined by the assumption of identical normal strains at the contact surface of stiffener and sheet.

The stability equations are solved for general instability under hydrostatic pressure by the method of virtual displacements. The solution used earlier for unstiffened conical shells, which satisfies some of the boundary conditions of simple supports only approximately, is again applied here. The effect of this incomplete compliance with boundary conditions is shown to be negligible by consideration of

"boundary work". The solution proposed for stiffened conical shells involves the concepts of "correcting coefficients" and minimization of corresponding "error loads".

Typical examples are analysed and the effect of eccentricity of stiffeners is investigated. Simplified approximate formulae for the critical pressure of frame-stiffened conical shells are also proposed.

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